

Anomaly Freedom of **BI-6**

The Modified Bianchi Identity (**BI-6**) for the Six-Force, Six-Dimensional Bulk (**GG-6**)

Katsushi Arisaka

*Department of Physics and Astronomy, Electrical and Computer Engineering,
University of California, Los Angeles, Los Angeles, CA 90095, USA*

Contact: arisaka@physics.ucla.edu

Abstract

For half a century a quantum anomaly has been described as a flaw that shows up only when a classical symmetry is run through the quantum machine. This paper makes the opposite case: the anomaly is, at bottom, a classical and geometric fact about a higher-dimensional space, and the quantum calculation merely *detects* it. The lineage is old — Bianchi wrote the geometric ancestor in 1902, and Green and Schwarz, in 1984, modified it for higher dimensions — and it is geometric at every step.

We show that the six-dimensional bulk of **GG-Theory**, built from the four axioms and seven postulates of its constitutional charter, passes the anomaly test completely. The proof has two halves: a local part that closes the usual one-loop consistency in a single line, and a global part that closes it under every large gauge transformation — with no restriction on the topology of the four-dimensional base, because the relevant spin bordism groups of the gauge group’s classifying space are shown to vanish. We also show that the five small integers (1, 3, 5; 7, 35) that fix every normalization in the theory are locked rather than fitted: the unified block size is forced by single-tensor consistency alone, the color index by the dual-lock closure together with the block embedding, and the remaining two by the same closure identity. No step of the cascade consumes a measured quantity.

Read alongside the companion papers, this places **BI-6** on the same foundational layer as the 1984 Green–Schwarz mechanism it generalizes — a program, begun with Riemann and Bianchi, of the geometrization of gauge in higher dimensions rather than the quantization of geometry. What makes a theory meaningful is not the cleverness of its calculations but the discipline of its consistency tests, and this is the sharpest one we know how to write down.

Executive Summary

An anomaly is a geometric fact, not a quantum accident. For more than a century the same idea has reappeared in new clothing. Bianchi’s 1902 identity (Bianchi, 1902) says that the curvature of a smooth space is “closed” — a purely geometric, pre-quantum statement; Yang and Mills (Yang and Mills, 1954) carried it over to the internal symmetries of the forces. What the Adler–Bell–Jackiw calculation of 1969 (Adler, 1969, Bell and Jackiw, 1969) found as a *quantum* anomaly, and what the descent formalism of Wess and Zumino (1971) and the modern differential-cohomology reading (Freed and Hopkins, 2021, Hopkins and Singer, 2005, Monnier and Moore, 2019) later clarified, is that the anomaly was geometric all along: a characteristic class on a bundle, classically defined and merely *detected* by the quantum loop rather than produced by it.

The bulk-side rewrite and its constitutional reading. Green and Schwarz (Green and Schwarz, 1984) showed in 1984 that this obstruction can be absorbed geometrically in a higher-dimensional bulk: replacing $dH = 0$ by $dH = X_4$ lets a bulk four-form flow in and cancel the projected boundary obstruction. In the reading of GG-A1-Constitution and GG-A2-GG6, that bulk identity is the content of AX-2 (GGEP), and the four-dimensional observer’s “anomaly” is its projected shadow. The companion paper GG-A2-GG6 (Arisaka, 2026, A2-GG6) derived the six-dimensional bulk of GG-Theory and stated its four-component modified Bianchi identity BI-6; the present paper proves that the resulting theory passes the anomaly test completely.

$$\boxed{dH_3 = X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - \frac{2}{5} F_B^2 - \frac{2}{5} F_{Qi}^2.} \quad (1)$$

The four-component identity (1) gathers the four curvature terms — gravity ($\text{Spin}(1, 5)$), the $\text{SU}(5)$ gauge force, and the two Abelian factors $\text{U}(1)_B$ and $\text{U}(1)_{Qi}$ — with the OGN-locked coefficients $(+1, -1, -2/5, -2/5)$ into a single closed gauge-invariant four-form X_4 , which inflows to the tensor field B_2 via $dH_3 = X_4$. The trace-ratio lock $|a/c| = |a/d| = 5/2$ pins both Abelian fractions symmetrically to $-2/5$; no coefficient is free to be adjusted.

Six principal results. The paper carries six principal results:

- **R1 — Rank-one criterion.** Single-tensor cancellation is possible if and only if the 4×4 anomaly matrix has rank one; the BI-6 matrix, built to the trace-ratio target and verified microscopically, saturates it.
- **R2 — Complete local anomaly cancellation.** Under the factorization $I_8 = \frac{1}{2} X_4^2$, the one-loop consistent anomaly is canceled exactly by the BI-6 tensor counterterm.
- **R3 — Complete global anomaly freedom.** On the physical intended GG-6 background class (GG-6: the six-dimensional Geometry-Gauge bulk geometry of the framework), the Wu–Chern–Simons / Hopkins–Singer / Monnier–Moore differential-cohomology framework’s four hypotheses (H1)–(H4) are verified automatically.

- **R4 — Constitutional admissibility of $U(1)_{Qi}$.** The GG-Theory Constitution admits both the three-component bulk $\text{Spin}(1, 5) \times \text{SU}(5) \times U(1)_B$ and the four-component bulk $\text{Spin}(1, 5) \times \text{SU}(5) \times U(1)_B \times U(1)_{Qi}$; the four-component version is adopted on empirical rather than constitutional grounds; the four-dimensional one-Abelian alternative (Benakli, Goodsell, and Goudeau, 2026) is engaged in Subsection 11.6.
- **R5 — Model-level benchmark spectrum.** An explicit finite chiral spectrum saturates the factorization, proving that the structural theorem is satisfiable; the gravitino-free (n_{SD}, n_F) closure is unique once the visible gauge content is fixed.
- **R6 — OGN-5 integer ledger as a theorem.** The integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ (OGN-5: the Occam-Gauge-Normalization five-tuple of POS-6) is locked by a five-step cascade internal to the BI-6 logic — three integers forced by prior constitutional theorems, two by the dual closure identity $K = LJ = M^2 + N^2 + L^2$ — including the trace-ratio identity $(L - N)/L = 2/5$ that bridges BI-6 to OGN-5.

What makes the proof distinctive. Three features set this proof apart from earlier six-dimensional anomaly work. It uses *no supersymmetry*: where prior constructions (Green, Schwarz, and West, 1985, Kumar, Morrison, and Taylor, 2010, Monnier and Moore, 2019, Sagnotti, 1992) lean on $\mathcal{N} = (1, 0)$ supergravity to constrain the spectrum, this proof leans on the constitutional clauses of GG-Theory (AX-1–AX-4, POS-1–POS-7) instead. The coefficient vector $(1, -1, -2/5, -2/5)$ is uniquely fixed by the trace-ratio lock and AX-4 (MSMF), with no continuous fitting freedom. And the same integers carry sharp predictions elsewhere in the cohort: the bioenergetic factor $Q_B \approx 0.1555$ eV per qubit matches ATP hydrolysis at biological temperatures (Bustamante *et al.*, 2005), the fine-structure value $\alpha^{-1}(0) = 2^J + N^2 = 137$ uses $J = 7$ and $N = 3$, and the kaxion haloscope window near 1.95 GHz is fenced by the throat depth set by the master lock integer $K = 35$ — all read directly off the ledger derived here.

Canonical implication chain. The present paper sits at the anomaly heart of the constitutional implication chain

$ \begin{aligned} \{\text{AX-1, AX-2, AX-3, AX-4}\} \cup \{\text{POS-1}, \dots, \text{POS-7}\} &\implies \{\text{GG-6 bulk, BI-6 identity}\} \\ &\implies \{\text{local cancellation, global completion, OGN-5 ledger}\} \\ &\implies \text{GDOS lawfulness} \\ &\implies \{\text{Gi-4} \oplus \text{Qi-4 pillars; SM; } \Lambda\text{CDM}\}. \end{aligned} $

(2)

The chain runs from the constitutional clauses, through the bulk theorems of GG-A2–GG6, through the local-plus-global anomaly proof of the present paper, into the lawfulness condition for the GDOS bulk-to-boundary projection, and finally into the boundary phenomenology. The present paper is the second descent step: local and global anomaly freedom is the gate that the third step (GDOS lawfulness) must clear before any boundary phenomenology can legitimately be invoked.

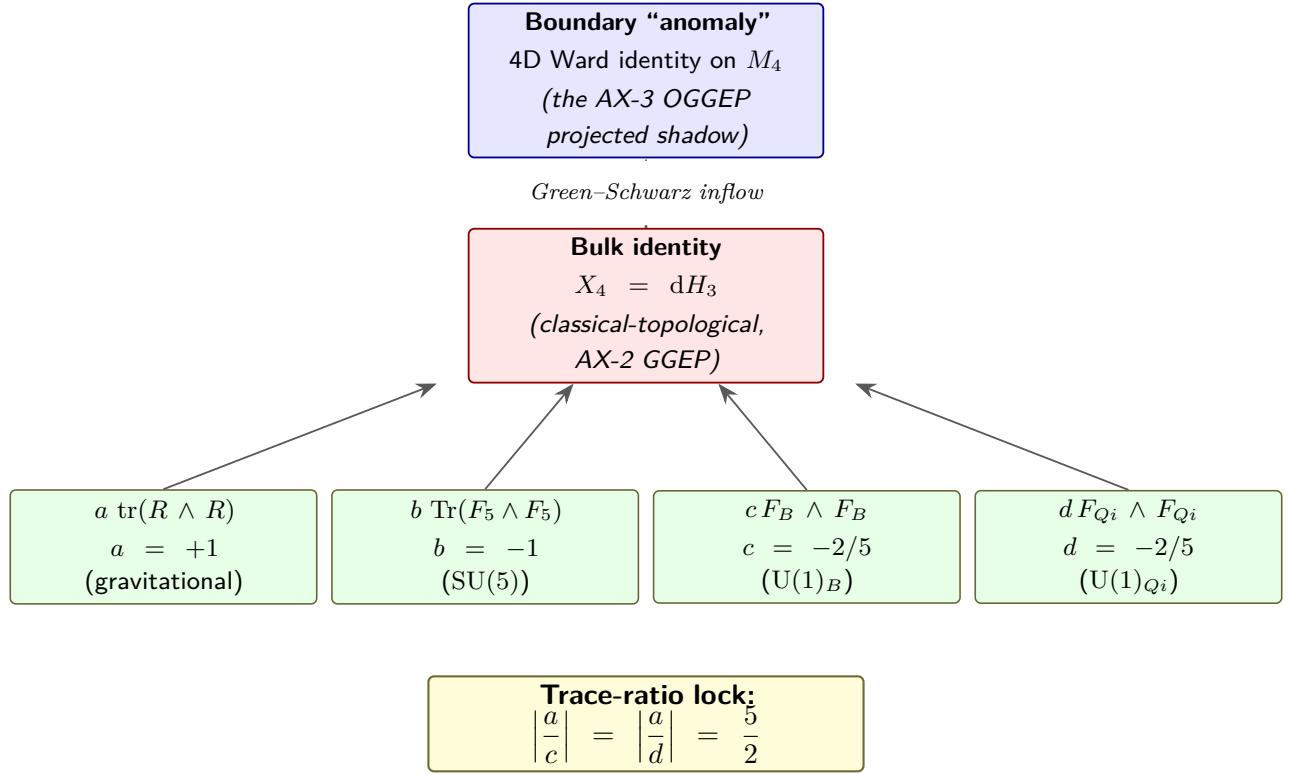
Similar mathematics, different physics. It is worth saying plainly how this construction stands relative to string theory, because the two share so much machinery. The modified Bianchi identity itself, the Green–Schwarz inflow that absorbs the anomaly, the differential-cohomology bookkeeping of Wu–Chern–Simons, and even the scale-profile metric with its logarithmic radial coordinate and bulk-to-boundary projection — all are borrowed, with gratitude, from the string program and its AdS/CFT and Randall–Sundrum descendants. What changes is the physics the mathematics is asked to carry. Where string constructions invoke supersymmetry and a large- N limit to control the spectrum, GG-Theory invokes its constitutional axioms and carries neither. Where the 10D Green–Schwarz polynomial factorizes through two functionally independent four-forms, the 6D BI-6 factorization is forced to be rank-one by a perfect-square condition, and is correspondingly more rigid. And where AdS/CFT reads the radial direction as an exact duality between a bulk gravity theory and a boundary conformal field theory, GG-6 reads it as a literal, irreversible renormalization-group scale interval under OGGEF. The 1984 move that Green and Schwarz introduced as a technical refinement of one specific model becomes, here, a foundational axiom (AX-2, GGEP) that any candidate theory of nature must satisfy. The same equations, in short, carrying a different claim about what they describe.

Philosophical implications. If the reading is correct, it carries a lesson that reaches past the anomaly. The deepest advances in physics have repeatedly turned out to be *recognitions* rather than inventions: Maxwell recognized that light was already present in his equations, and Einstein that gravity was already present in Riemann’s geometry. This paper proposes that the chiral anomaly is the next such recognition — not a defect of quantum field theory to be exorcised, but a classical, topological identity on a higher-dimensional bulk, of which the four-dimensional “anomaly” is only the shadow cast on the observer’s boundary. Read this way, the program is not the quantization of geometry that has organized so much of the last half-century, but its mirror image: the geometrization of gauge. The discipline this demands — a theory with zero free continuous parameters, every number forced by an integer ledger rather than fitted to the data — is itself a philosophical commitment: that a law of nature should predict the world rather than be made to accommodate it. Whether nature honors it is an empirical question, not a philosophical one. Its sharpest form — one value for each low-energy constant across the whole sky — is the *second horizon problem* of GG-A10-Cosmos and GG-P3-Electron.

Closing reflection. The test of an honest framework is not that it answers every question on the first pass, but that the answers it does give are sharp enough to be wrong. The local and global anomaly freedom theorem is exactly such an answer: if a single anomaly term is ever found that escapes the four-component lock, the whole construction is refuted upstream. The proposal will be vindicated or refuted by quantitative tests over the 2026–2040 horizon: the KK lock cluster near $M_{\text{lock}} \approx 3.85$ TeV, the kaxion signal near ~ 8.1 μeV , and the cosmological inventory $\Omega_m = 1/\pi$. Until then, the recognition is a proposal, not a confirmed discovery. That is the bargain we offer.

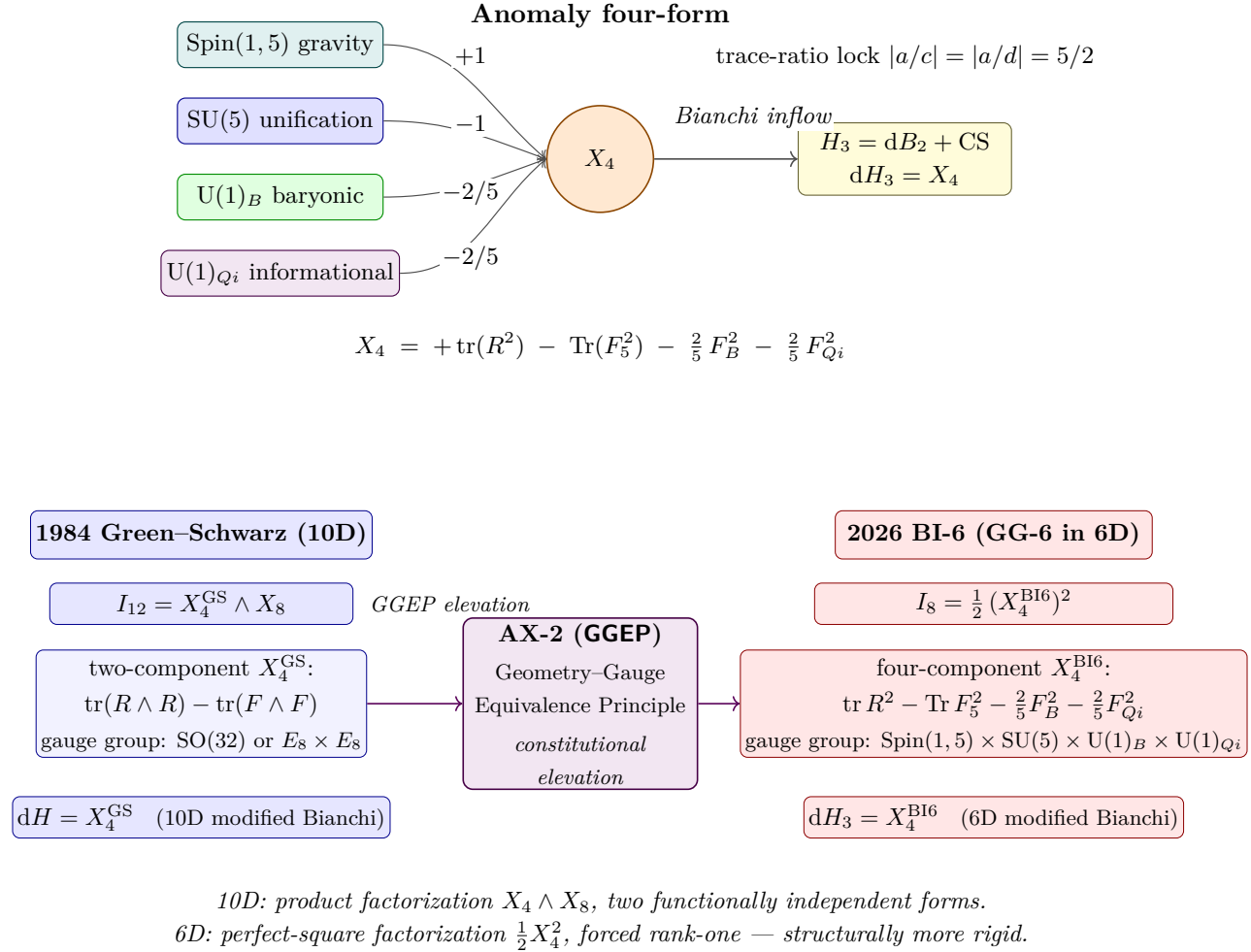
Keywords: GG-Theory; BI-6 anomaly identity; local and global anomaly cancellation; Wu–Chern–Simons differential cohomology; Green–Schwarz mechanism; rank-one factorization; trace-ratio lock; OGN-5 integer ledger; falsifiability.

The Anomaly Is Not a Defect of Quantum Field Theory. It Is a Classical Identity, One Dimension Up



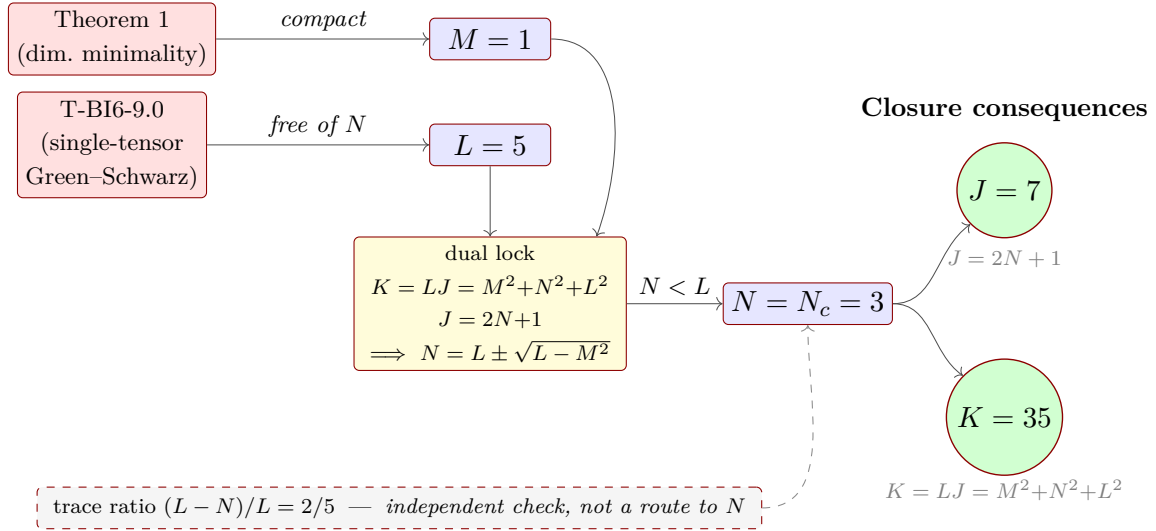
The four-component BI-6 bulk-side classical-topological identity and its boundary shadow. The four curvature contributions (green) carry the canonical coefficients $(a, b, c, d) = (1, -1, -2/5, -2/5)$ and assemble into the single closed gauge-invariant four-form X_4 (red), a classical identity on the AX-2 GGEF composite connection, pinned by the trace-ratio lock $|a/c| = |a/d| = 5/2$ (yellow). The boundary “anomaly” (blue) is the AX-3 OGGEF projected shadow of this bulk identity — a boundary re-measurement of a classical fact already in force in the bulk, not a quantum pathology to be cured. Full version, with the descent proof it summarizes: Figure 1 in Part I, p. 31.

Four Curvature Squares, One Closed Form — and the 1984 Mechanism Raised to an Axiom



What the four components are, and where they came from. *Above:* the bulk gauge–gravity group Spin(1, 5) $\times$ SU(5) $\times$ U(1)_B $\times$ U(1)_{Qi} contributes four distinct curvature-squared four-forms, which combine with the OGN-5-locked coefficients (+1, −1, −2/5, −2/5) into the single closed gauge-invariant X_4 ; the trace-ratio lock $|a/c| = |a/d| = 5/2$ pins the Abelian fraction, and the modified Bianchi identity $dH_3 = X_4$ closes the whole inflow through one tensor sector. *Below:* the same object read against the mechanism it generalizes. Green and Schwarz pinned a *two-component* pattern in ten dimensions in 1984, as a technical refinement realized in a specific model; BI-6 is the four-component version, and what GG-Theory adds is the axiom – AX-2 (GGEP) – that raises the move from a refinement to a requirement any candidate theory must meet. The 10D pattern admits two functionally independent forms; the 6D pattern is rank-one by perfect-square forcing, and so is the more rigid of the two. Both panels appear again at full size, with their proofs, as Figure 4 in Part II and Figure 7 in Part IV.

Five Integers, Forced — and No Free Parameter Left in the Bulk



The OGN-5 Diophantine cascade. Two theorems force $(M, L) = (1, 5)$ without reference to any color count (left column); the dual lock together with the block embedding $N < L$ then determines $N = N_c = 3$, and the pair $(J, K) = (7, 35)$ follows from the closure-index clause and the identity $K = LJ = M^2 + N^2 + L^2 = 35$ (right column). The double equality $35 = 5 \times 7 = 1 + 9 + 25$ is the axiomatic fingerprint of OGN-5: it is not a coincidence but the content of the dual lock, and every numerical prediction of the program descends from these five integers. Solid arrows mark the forcing chain; the one dashed arrow carries the check. After the cascade closes, no continuous dimensionless parameter survives in the bulk theory. Full version, and the proof of each of the five steps: Figure 6 in Part III, p. 99.

What Is Actually Proven, Level by Level — Against the 1984 Mechanism This Descends From

What is actually proven, at each level of the argument. The rows run from the structural theorem down to the specific model, and each cell records a status rather than a claim of superiority. The reading is mixed: the ten-dimensional structural theorem has been proved since 1984 and is not in question here, while the construction in this paper is younger and proves less at the model level. The table is here so that neither fact has to be inferred. Read with its surrounding assessment in Table 8, p. 106.

Level of proof	10D superstring	6D BI-6 / GG-6
Structural GS theorem	Proven (1984)	Proven (this paper)
Coefficient uniqueness	Proven (1/30 for $SO(32)$)	Proven (2/5 from OGN)
Rank-one / factorization	Implicit	Explicit (Thm. 5.1)
BRST / Slavnov–Taylor	Proven	Proven (Thm. 5.5)
Global gauge anomaly	Proven	Proven ($\pi_6 = 0$)
Explicit spectrum	Known (Type I / heterotic)	Benchmark constructed (Thm. 7.9)
Minimal spectrum	Known (SUSY fixes it)	Open (finite Diophantine search)
Gravitational completion	Automatic (SUGRA)	Explicit (SD tensors)
Charge lattice	String charge quantization	Abelian rescaling
Landscape / vacuum counting	$\sim 10^{500}$	Finite (bounded)

Every Claim in This Paper, with Its Status — Go to the Weakest Row

Every principal claim of this paper, with its status and a pointer. The middle column is graded rather than rhetorical: a claim is proved, conditional on a stated hypothesis, or open, and the third column says where the proof lives or what the residue is. A reader who wants to test the paper should read this table first and then go to the weakest row, because the paper is only as strong as that row. Nothing downstream in the cohort consumes a row that is not marked proved. The same table, in place: Table 2, p. 24.

Claim	Status	Comment
If $I_8 = \frac{1}{2}X_4^2$, a single Green–Schwarz tensor cancels the full infinitesimal local anomaly	Proved	Theorem 5.5; rigorous descent proof in Section 5 and Appendix B.
The same BI-6 mechanism is globally anomaly-free on the intended GG-6 background class	Proved (two levels)	Section 6 and Appendix I; the theorem is unconditional for the intended product/collar GG-6 category.
An explicit finite benchmark spectrum realizes $I_8 = (\Omega/2)X_4^2$ for BI-6	Proved	Theorem 7.9; existence construction, not a uniqueness or minimality claim.
The same BI-6 term cancels orbifold-localized boundary anomaly by inflow	Proved	Standard descent on the interval, together with the globally defined differential-cocycle lift.
The OGN-5 integers $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ are theorem-level consequences of GG-6 with BI-6	Proved, given AX-4	Section 9; the primitive tensor slot, the unified block size (Thm. 9.2), the closure determination of the color index (Thm. 9.4) and the dual lock fix the packet uniquely, with no empirical input. The fence is AX-4 (MSMF), which selects the minimal package at two steps.
$U(1)_B$ is essential to the BI-6 / GG-6 chain, while $U(1)_{Q_i}$ is allowed and optional at the narrow anomaly-theorem level	Proved / clarified	Section 6; the baryonic slot is structural, the informational slot is anomaly-compatible and becomes natural/constitutive once GG-6 is extended to GDOS.
BI-6 is analogous to the 1984 GS mechanism at the level of local descent/cohomology and now also admits a modern global completion	Proved	Sections 10 and 11; similarities and differences are stated feature by feature.
The benchmark spectrum is unique or absolutely minimal	Open	The paper gives an existence construction; minimality is a separate Diophantine problem.
$\Omega_{\text{eff}} = \Omega_{\text{orb}}/L^2$ is driven dynamically to 1 by the inflationary descent of GG-A10-Cosmos	Deferred	Demonstrated in the companion cosmology paper.
The 28 Standard-Model parameters are derived from OGN-5 / GDOS	Deferred	Demonstrated in the companion Arisaka-GG-A9-SM (Arisaka, 2026, A9-SM) rather than proved in the present preprint.

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Notation

A handful of names and one typographic convention appear in the Abstract, the Executive Summary and the Preface before the text has a chance to define them. They are collected here, one line each, so that nothing in the opening pages has to be taken on faith. The full dictionary — every abbreviation in the paper, with the companion paper that introduces it — is Appendix A.

The two traces. Three traces appear in this paper and are distinguished typographically. $\text{tr } R^2$ is the trace over the six-dimensional tangent indices of the Lorentz curvature two-form. $\text{Tr } F_5^2$ is the trace over the *fundamental* **5** of $\text{SU}(5)$; where a different representation is meant it carries an explicit subscript. F_B^2 and $F_{Q_i}^2$ carry no trace at all: an Abelian field strength is a one-by-one matrix. (Appendix C follows the ten-dimensional literature’s convention for comparison and writes adjoint traces explicitly as such; no adjoint trace appears anywhere in the six-dimensional argument.)

One symbol, four meanings. The letter Ω is overloaded in this literature and this paper does not escape it. Four distinct objects wear it, and three of them appear within four pages of each other in Part II. Ω_{orb} (written Ω where no confusion can arise) is the Green–Schwarz reservoir normalization, an integer in $25\mathbb{Z}$. $\Omega_{\text{eff}} = \Omega_{\text{orb}}/L^2$ is the late-time effective normalization, which the cosmological descent drives to one. Ω_q^{Spin} with a superscript is a spin bordism group. And Ω_m , Ω_b , Ω_Λ with a subscripted Latin or Greek letter are cosmological density parameters. The subscript always disambiguates; where the bare symbol appears it means the reservoir normalization.

The names used before they are defined.

Table 1: The nine names this paper uses before it defines them, gathered so that a reader can begin at Part I without a companion paper open. Each entry says what the object is and which paper fixes it. Nothing here is defined — every row is a pointer, and the definitions live where the pointers say they do. The order is the order in which the argument reaches them, not the alphabet.

GG-Theory	The twenty-two-paper program this paper belongs to; its axioms and postulates are fixed in GG-A1-Constitution .
GG-6	Geometry–Gauge locking in six dimensions: the bulk this paper tests, derived in GG-A2-GG6 .
BI-6	The four-component modified Bianchi identity $dH_3 = X_4$ of that bulk — the object whose consistency is the subject of this paper.
OGN-5	The five-integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$, derived in Part III.
AX 1–4	The four axioms of the Constitution: locality and causality; geometry–gauge equivalence (GGEP); its operational extension (OGGEP); and maximum symmetry, minimum freedom (MSMF).
POS 1–7	The seven postulates: gauge invariance across scales, the unified extremal principle, anomaly freedom, closure and positivity, renormalization-group covariance, Occam gauge normalization, and measurement and falsifiability.
GDOS	Geometric Dynamics of Open Systems — the bulk-to-boundary projection carried by GG-A5-GDOS , which consumes this paper’s global theorem.
Gi-4, Qi-4	The geometric and quantum-information pillars of GG-A6-GRIP and GG-A7-Qi4 .
BRST	The algebraic framework, named for its four originators, in which “the anomaly cancels exactly” is given its precise form in Part II.

Every other abbreviation in this paper is expanded where it first appears and again in Appendix A.

Preface

In 1969, three independent papers by Stephen Adler, John Bell, and Roman Jackiw made a strange and beautiful discovery: a symmetry of a classical field theory could fail at the quantum level. The classical equations of motion conserved a current; the quantum effective action did not. What looked like an exact conservation law was, in fact, broken by a tiny topological term. The discovery was called the chiral anomaly, and it has been one of the deepest themes of twentieth-century physics ever since.

The chiral anomaly is not always a bug. Sometimes it is a feature: it explains the observed decay rate of the neutral pion, for example, with no free parameters. But in any theory of fundamental physics, the anomaly has to be tamed. If the symmetry being broken is the gauge symmetry that defines the theory, the whole theory falls apart. The standard fix is to arrange the fermion content so that the anomaly cancels exactly — a delicate balance that has constrained the Standard Model from the beginning. In 1984, Michael Green and John Schwarz discovered a more elegant alternative: in higher-dimensional theories, the anomaly can be canceled by a topological inflow from a bulk field. The construction worked beautifully in ten-dimensional string theory and became the technical heart of the first superstring revolution.

This paper proposes that the deepest content of the Green–Schwarz construction is not what it does for string theory but what it does for the relationship between geometry and gauge. The modified Bianchi identity that Green and Schwarz wrote down is a geometric statement, classical and topological, that says gravity and gauge curvature must live inside one closed topological four-form on the bulk manifold. Read carefully, it is the equivalence of geometry and gauge in higher dimensions. We extend the construction to six dimensions with four-component anomaly inflow, prove that it admits local and global anomaly freedom on the physical intended bulk class, and show that the integer ledger of the entire program is a Diophantine consequence of the same identity.

We invite the reader to walk the proof, and to read the long supportive treatment in the final section as an invitation rather than a critique. The Green–Schwarz mechanism was the moment string theory glimpsed that geometry and gauge belong in the same picture. The proposal here is that the picture has a second home — in a six-dimensional bulk that is small enough to audit and rigid enough to predict.

Part I

Introduction

Background, motivation, and the anomaly problem in six dimensions

The first two papers of this series established the axiomatic framework (Arisaka-GG-A1-Constitution ([Arisaka, 2026, A1-Constitution](#))) and the six-dimensional bulk with its gauge-gravity structure and modified Bianchi identity (Arisaka-GG-A2-GG6 ([Arisaka, 2026, A2-GG6](#))). The present paper asks the next unavoidable question: *does that six-dimensional stage define a mathematically consistent quantum theory?* The four sections in this Part set the stage.

Here is the problem in plain terms. A gauge theory is a theory built on a redundancy: the mathematical description carries more information than the physics does, and the physics is whatever survives when the surplus is thrown away. That works only if the surplus really is surplus. An *anomaly* is what happens when a symmetry that is exact in the classical description fails once the theory is quantized. If the failing symmetry is a decorative one, the failure is a physical effect and can be measured; the decay of the neutral pion is the textbook case. If the failing symmetry is the gauge redundancy itself, there is no theory left. Probabilities stop summing to one, states of negative norm appear in the spectrum, and no prediction the theory makes can be trusted. This is not a defect to be repaired later. It is the entrance examination every candidate theory of nature has to pass before any of its other claims are worth discussing.

Six dimensions make the examination harder than four, not easier. The anomaly lives in a polynomial two degrees higher than the spacetime dimension, so in six dimensions it is an eight-form, and it contains a term that no rearrangement of the matter content can cancel against another. Green and Schwarz found the way out in 1984, in ten dimensions: if the dangerous polynomial happens to factorize, a single tensor field can be made to cancel it by flowing charge in from the bulk. The price is that the factorization is a very strong condition, and almost no theory satisfies it.

Sections [1](#) and [2](#) explain what an anomaly is and trace the idea back to its geometric origin, from Bianchi’s 1902 identity to the differential-cohomology language of the last two decades. Section [3](#) states the Green–Schwarz mechanism in general even dimensions, so that the six-dimensional case that follows is a specialization of something already in hand rather than a trick. Section [4](#) says what this paper inherits from its two predecessors, what the rest of the cohort is waiting on, and how the remaining nine Parts are arranged.

1 Introduction

1.1 The anomaly problem in higher-dimensional gauge theories

Quantum anomalies are the sharpest consistency constraints in theoretical physics. A chiral gauge theory whose quantum effective action fails to be gauge invariant is not merely inaccurate; it

is *inconsistent*: unphysical gauge modes fail to decouple, Ward identities are violated, and unitarity is lost. In even spacetime dimensions, the local anomaly is encoded by an anomaly polynomial of degree $d + 2$, where d is the spacetime dimension. Cancellation of this polynomial is therefore a *necessary condition* for any consistent chiral gauge theory.

The anomaly problem becomes especially acute in higher dimensions. In four dimensions, the Standard Model cancels its anomalies through a representation-theoretic miracle: the specific quantum numbers of quarks and leptons conspire to make all gauge and mixed anomaly coefficients vanish identically. No additional mechanism is needed. In six and ten dimensions, however, the anomaly polynomial is richer (an eight-form or twelve-form, respectively), and representation-theoretic cancellation alone is generically insufficient. A new mechanism is required: the **Green–Schwarz mechanism** (Green and Schwarz, 1984), which introduces a higher-form tensor field whose classical variation absorbs the quantum anomaly.

The discovery of this mechanism in 1984 by Green and Schwarz for the ten-dimensional superstring was one of the pivotal events in modern theoretical physics. It showed that anomaly cancellation in ten dimensions is possible for exactly two gauge groups— $SO(32)$ and $E_8 \times E_8$ —and launched the “first superstring revolution.”

The present paper develops the six-dimensional counterpart of the Green–Schwarz mechanism within a fundamentally different theoretical framework: **GG-Theory**, which derives anomaly cancellation not from supersymmetry but from a constitutional axiom system.

1.2 The GG-Theory Constitution

The GG-Theory program, developed in the companion paper (Arisaka, 2026, A1-Constitution), proposes that the foundational layer of physics is not a specific Lagrangian but a **constitution**: a compact set of admissibility rules that any consistent theory must satisfy. The constitution consists of four axioms and seven postulates:

- **AX-1** (LC): Locality and Causality—admissible evolution is local and causal; no acausal effective kernels are allowed.
- **AX-2** (GGEP): Geometry–Gauge Equivalence Principle—the primitive field is a composite geometry $\oplus$ gauge connection; gravity and gauge structure are not separate fundamental sectors.
- **AX-3** (OGGEP): Operational/Open Geometry–Gauge Equivalence Principle—a bulk theory is incomplete unless it supplies a lawful bulk-to-boundary registration map producing operational observables.
- **AX-4** (MSMF): Maximum Symmetry, Minimum Freedom—among admissible theories, nature selects those with maximal symmetry and minimal non-forced continuous freedom.

The seven postulates (POS-1 through POS-7) refine admissibility through gauge invariance, a unified extremal principle, anomaly-free admissibility by inflow, closure and positivity, RG compatibility, Occam Gauge Normalization, and measurement/falsifiability.

1.3 GG-6 and the BI-6 identity

Historical background (with citations). Generally speaking, the identity $dH_3 = X_4$ that we are about to write down is a particular *modified* instance of a much older geometric identity. In a 1902 note in the *Atti dell’Accademia dei Lincei*, Luigi Bianchi (Bianchi, 1902) proved that the Riemann curvature tensor satisfies the differential identity $D_{[\mu}R_{\nu\rho]\sigma\tau} = 0$ — equivalently, $DR = 0$ when R is read as a curvature two-form on the frame bundle. The Bianchi identity is the geometric content of the statement that $D^2 = 0$ on a connection, and it is the classical reason every gauge field strength F obeys $dF = 0$ on a principal bundle. The Green–Schwarz mechanism (Green and Schwarz, 1984) of Section 3 introduces a controlled, geometric *modification* of the Bianchi identity for the antisymmetric tensor field strength H_3 , replacing $dH_3 = 0$ by $dH_3 = X_4$ for a closed, gauge-invariant, characteristic-class four-form X_4 built from $\text{tr}(R \wedge R)$ and $\text{tr}(F \wedge F)$. The Green–Schwarz move is, in this geometric reading, simply the statement that the natural Bianchi identity for the tensor sector of a modern higher-dimensional bulk theory is not $dH = 0$ but $dH = X_4$ — a closed gauge-invariant four-form built from the curvature and the gauge field strengths. BI-6 is the four-component generalization of that modified-Bianchi pattern, with the four contributors (R, F_5, F_B, F_{Qi}) pinned by the OGN-5 ledger and the constitutional axioms of GG-Theory; we spell out the construction immediately below.

The companion paper (Arisaka, 2026, A2-GG6) derives, from AX-1–AX-4 and POS-1–POS-7, a minimal six-dimensional bulk

$$M_6 = M_4 \times I_r \times S_\theta^1, \quad r = \ln(E_{\text{GUT}}/\mu), \quad (3)$$

with composite gauge–gravity group

$$G_{\text{Bulk}} = \text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Qi}, \quad (4)$$

where the Lorentz spin factor $\text{Spin}(1, 5)$ enters on equal logical footing with the three internal Lie-group factors via AX-2 (GGEP), replacing the unstructured “ $\times$ Gravity” label of the legacy notation. The canonical bulk metric is

$$ds_6^2 = e^{-2r} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2 + d\theta^2, \quad r \in [0, K], \quad \theta \in [0, 2\pi), \quad (5)$$

with $\eta_{\mu\nu}$ the Minkowski metric on the 4D boundary M_4 , r the natural-log RG-flow coordinate from the UV boundary at $r = 0$ ($\mu = E_{\text{GUT}}$) to the IR boundary at $r = K$ ($\mu = E_{\text{GUT}} e^{-K}$), and θ a dimensionless topological circle that hosts integer-quantized Wilson-line phases.

The unique geometrical structure of GG-6: a minimalist’s 6D bulk. The canonical metric (5) is what one obtains when AX-4 (MSMF), POS-5 (RG), and POS-6 (OGN) are demanded simultaneously of a six-dimensional bulk: the four-dimensional piece is flat-Minkowski with a single exponential scale profile e^{-2r} locked to the natural RG-scale parametrization $r = \ln(E_{\text{GUT}}/\mu)$; the radial fiber I_r is logarithmic in the boundary energy scale rather than a free spatial dimension; and the topological circle S_θ^1 has no intrinsic length, $G_{\theta\theta} = 1$ in natural GG-6-units. In the cohort’s

vocabulary the radial fiber I_r is the *scale dimension* and the circle S_θ^1 is the *gauge dimension*; the two are collectively the *non-spatial dimensions* of the bulk, the term *extra dimension* being reserved here for the *spatial* Kaluza–Klein and string directions against which GG-6 is compared. The four conventional continuous parameters of a generic Randall–Sundrum-style construction — the AdS curvature scale k , the compact-circle radius R_θ , the scale endpoint r_c , and the brane tension σ — are all absent: $k = 1$ is the natural-RG-scale normalization forced by POS-6, R_θ is a unit-conversion artifact that disappears in natural units, r_c is replaced by the integer endpoint K of the OGN-5 ledger, and the brane tension is fixed by the bulk-curvature condition. What remains is a minimalist’s 6D bulk with *no continuous geometric parameters at all*: every dimensionless quantity in the metric is either zero, one, or the single integer $K = 35$ that the OGN-5 cascade fixes in the companion paper (Arisaka, 2026, A2-GG6). This is the geometric reason the six-dimensional bulk passes the AX-4 (MSMF) test: the conventional KK–RS continuous parameter landscape of dimensions, radii, and curvature scales has been collapsed entirely to a discrete arithmetic ledger. In the language of the companion paper, this scale-profile metric is the renormalization-group geometrization of the renormalization-group flow — radial evolution *is* the running of couplings, not motion through an extra spatial dimension — so the construction is read on the field-theory side of the AdS/CFT dictionary rather than as a Randall–Sundrum spatial braneworld (Arisaka, 2026, A2-GG6).

The Green–Schwarz-modified Bianchi identity BI-6 is

$$\boxed{dH_3 = X_4 = \text{tr}(R \wedge R) - \text{Tr}(F_5 \wedge F_5) - \frac{2}{5} F_B \wedge F_B - \frac{2}{5} F_{Qi} \wedge F_{Qi}.} \quad (6)$$

The coefficient vector $(a, b, c, d) = (1, -1, -2/5, -2/5)$ is derived from the trace ratio of the primitive baryon and hypercharge generators in the fundamental **5** of SU(5). The OGN-5 integer ledger

$$(M, N, L; J, K) = (1, 3, 5; 7, 35) \quad (7)$$

compresses all dimensionless normalization freedom (beyond Newton’s constant G_N) into a discrete arithmetic package.

1.4 This paper as the direct follow-up to GG-A1-Constitution and GG-A2-GG6

Arisaka-GG-A1-Constitution (Arisaka, 2026, A1-Constitution) proposes the constitution; Arisaka-GG-A2-GG6 (Arisaka, 2026, A2-GG6) derives the six-dimensional stage, the composite gauge–gravity packet, and BI-6; the present paper determines whether that stage is *actually admissible as a quantum theory*. In that sense, this is not an auxiliary technical supplement but the first decisive consistency test of GG-6. A six-dimensional bulk that fails anomaly freedom cannot serve as the parent of any later application in the series. Conversely, once anomaly freedom is established, the later uses of OGGEF and GDOS are placed on a much firmer mathematical foundation.

The paper is therefore two-layered. The main text states the theorem package in publishable form and develops the physical interpretation. The appendices then provide the full descent proof, the exact coefficient-matching equations, the detailed orbit solution, the global mapping-torus

argument, and the automatic-admissibility lemmas. The standard is Euclidean: each step must follow from the constitutional and GG-6 premises.

1.5 Relation to GDOS and the open-system boundary

The companion paper (Arisaka, 2026, A5-GDOS) extends the GG-6 framework to open systems through the Geometry-Dynamics-Open-System (GDOS) construction. While GG-6 establishes the six-dimensional bulk theory, GDOS derives the four-dimensional boundary theory as the operational shadow of the bulk: quantum mechanics, measurement, and the Nakajima–Zwanzig open-system evolution (Nakajima, 1958, Zwanzig, 1960) emerge as boundary projections of the closed bulk dynamics through the AX-3 (OGGEP) channel.

The anomaly cancellation proved in the present paper is a *prerequisite* for GDOS. If the six-dimensional bulk were anomalous, the bulk-to-boundary channel would be ill-defined: the Ward identities that protect the boundary observables would fail, and the operational dictionary would not be lawful. Conversely, the boundary anomaly generated by the orbifold projection (which selects the Standard-Model zero modes) is exactly canceled by inflow from the same BI-6 term.

Thus the anomaly proof of the present paper is the *bridge* between the bulk theorem of Arisaka-GG-A2-GG6 (Arisaka, 2026, A2-GG6) and the boundary theorem of the GDOS paper (Arisaka, 2026, A5-GDOS).

1.6 What is proved here and what now remains open

Table 2 summarizes the status of the main claims. It separates theorem-level statements that are now closed from statements that remain conditional, deferred, or open. The most important shift in the logical structure of the series is the promotion of BI-6 from a theory with only local cancellation plus preliminary bordism checks to a theory with a complete local-and-global anomaly theorem on the intended GG-6 background class.

Table 2: Every principal claim of this paper with its status and a pointer. The middle column is graded rather than rhetorical: a claim is proved, conditional on a stated hypothesis, or open, and the third column says where the proof lives or what the residue is. A reader who wants to test the paper should read this table first and then go to the weakest row, because the paper is only as strong as that row. Nothing downstream in the cohort consumes a row that is not marked proved.

Claim	Status	Comment
If $I_8 = \frac{1}{2}X_4^2$, a single Green–Schwarz tensor cancels the full infinitesimal local anomaly	Proved	Theorem 5.5; rigorous descent proof in Section 5 and Appendix B.
The same BI-6 mechanism is globally anomaly-free on the intended GG-6 background class	Proved (two levels)	Section 6 and Appendix I; the theorem is unconditional for the intended product/collar GG-6 category.
An explicit finite benchmark spectrum realizes $I_8 = (\Omega/2)X_4^2$ for BI-6	Proved	Theorem 7.9; existence construction, not a uniqueness or minimality claim.
The same BI-6 term cancels orbifold-localized boundary anomaly by inflow	Proved	Standard descent on the interval, together with the globally defined differential-cocycle lift.
The OGN-5 integers $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ are theorem-level consequences of GG-6 with BI-6	Proved, given AX-4	Section 9; the primitive tensor slot, the unified block size (Thm. 9.2), the closure determination of the color index (Thm. 9.4) and the dual lock fix the packet uniquely, with no empirical input. The fence is AX-4 (MSMF), which selects the minimal package at two steps.
$U(1)_B$ is essential to the BI-6 / GG-6 chain, while $U(1)_{Q_i}$ is allowed and optional at the narrow anomaly-theorem level	Proved / clarified	Section 6; the baryonic slot is structural, the informational slot is anomaly-compatible and becomes natural/constitutive once GG-6 is extended to GDOS.
BI-6 is analogous to the 1984 GS mechanism at the level of local descent/cohomology and now also admits a modern global completion	Proved	Sections 10 and 11; similarities and differences are stated feature by feature.
The benchmark spectrum is unique or absolutely minimal	Open	The paper gives an existence construction; minimality is a separate Diophantine problem.
$\Omega_{\text{eff}} = \Omega_{\text{orb}}/L^2$ is driven dynamically to 1 by the inflationary descent of GG-A10-Cosmos	Deferred	Demonstrated in the companion cosmology paper.
The 28 Standard-Model parameters are derived from OGN-5 / GDOS	Deferred	Demonstrated in the companion Arisaka-GG-A9-SM (Arisaka, 2026, A9-SM) rather than proved in the present preprint.

2 The geometric origin of the anomaly concept: a constitutional reading of a hundred and twenty years

The conventional reading of the gauge anomaly — that it is a modern quantum-mechanical effect, born in 1969 with the Adler–Bell–Jackiw triangle calculation and tamed only when the 1980s anomaly-polynomial machinery matured — has been the dominant narrative for a half-century. In this section we propose a different reading. The fundamental object underlying the anomaly is not

a quantum loop but a classical, topological, differential-geometric identity on a higher-dimensional bulk manifold. Quantum mechanics *detects* the identity at the boundary; it does not create the identity. The Bianchi identity that Luigi Bianchi published in 1902 is the geometric ancestor; the Green–Schwarz modification of 1984 is the bulk-side rewrite that makes the boundary projection self-consistent in higher dimensions; the modern Hopkins–Singer differential-cohomology framework is the formal classification of the topological object. Under the constitutional clauses of GG-A1-Constitution (Arisaka, 2026, A1-Constitution), the bulk identity is forced by AX-2 (GGEP); the boundary observer’s “anomaly” is the AX-3 (OGGEP) projected shadow of the same identity. This reading reframes the entire program of anomaly cancellation as *geometric all the way down*, and we develop it as the foundational reading on which the BI-6 theorem of the present paper rests.

2.1 Bianchi (1902): pure geometry, no quantum content

Luigi Bianchi’s note *Sui simboli a quattro indici e sulla curvatura di Riemann* (Bianchi, 1902), published in the *Atti della Reale Accademia dei Lincei* in 1902, proves the differential identity

$$D_{[\mu}R_{\nu\rho]\sigma\tau} = 0, \quad (8)$$

where R is the Riemann curvature tensor of a smooth Riemannian manifold and D is the metric-compatible covariant derivative. Equivalently, reading R as a curvature two-form on the frame bundle, the identity is the closed-form statement

$$DR = 0 \quad (9)$$

under the covariant exterior derivative. The mathematical content is purely geometric: it says that the curvature two-form of a smooth connection on a principal bundle is closed under the covariant exterior derivative. This is the geometric content of the operator identity $D^2 = 0$ acting on a connection one-form A , with $R = dA + A \wedge A$. In plain language, it asserts that the underlying manifold is smoothly connected — every point joins to its neighbors through a single, well-defined connection — and that any closed flow of conserved charge is continuous. No physics enters Bianchi’s note. No measurement is performed. No quantum loop is computed. The identity is a property of the bundle, not of any process running on it. Riemann’s 1854 *Habilitationsschrift* (Riemann, 1854) had developed the manifold language fifty years earlier; Bianchi’s 1902 note extracted the differential consequence in the form that the twentieth-century gauge theories would inherit.

2.2 Yang–Mills (1954): the gauge-bundle Bianchi

The Yang–Mills construction (Yang and Mills, 1954) extends the Bianchi pattern from the frame bundle of spacetime to a principal G bundle of internal symmetry. The gauge field strength $F = dA + A \wedge A$ of any smooth connection A on a G bundle satisfies the classical Bianchi identity

$$DF = dF + [A, F] = 0, \quad (10)$$

where D is the gauge-covariant exterior derivative. This is again pure geometry: it is the bundle-theoretic statement that $D^2 = 0$ on the gauge connection. Yang and Mills wrote it down in 1954 in the context of isotopic-spin gauge invariance, not as a quantum-mechanical statement; the identity is true for any smooth gauge field configuration on any principal bundle, classical or quantum, perturbative or non-perturbative. The gauge anomalies that the next generation of theorists would struggle with arise on top of this identity, not in violation of it.

2.3 Adler–Bell–Jackiw (1969): the quantum window onto a classical fact

The one-loop triangle calculation of [Adler \(1969\)](#) and [Bell and Jackiw \(1969\)](#) in QED with chiral fermions produced the celebrated result

$$\partial_\mu J^{5\mu} = \frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (11)$$

showing that the classical chiral-current conservation law fails at one loop in the presence of a background gauge field. Read through the conventional lens, this looks like a defining example of a quantum-mechanical effect: the classical Noether symmetry was exact, the quantum theory broke it, and the “anomaly” was named to mark the breakdown. We propose a different reading, supported by the entire structural development of anomaly theory between 1971 and the present. The right-hand side of (11), $F \wedge \tilde{F} = \text{tr}(F \wedge F)$, is a *characteristic class* on the gauge bundle — specifically, the second Chern class up to normalization. Characteristic classes are classical topological invariants of the bundle, fully defined and calculable from any smooth connection, with no reference to quantum dynamics. The triangle calculation does not *produce* this characteristic class; it *measures* it, through the quantum amplitude that happens to be sensitive to the underlying topology of the bundle. The “anomaly” the quantum calculation reveals is the boundary-observer’s first sight of a classical topological object that was present in the bundle all along. In the constitutional language, the ABJ calculation is the first historical opening of a window onto the AX-2 (GGEP) content of the bundle; the constitutional reading was identified later but the content was always classical-geometric.

2.4 Wess–Zumino consistency (1971) and the cohomological reading

[Wess and Zumino \(1971\)](#) reformulated anomaly cancellation as a cohomological consistency condition: the anomaly $\mathcal{A}[g]$ as a functional of the gauge parameter g must satisfy

$$\delta_{g_2} \mathcal{A}[g_1] - \delta_{g_1} \mathcal{A}[g_2] = \mathcal{A}[[g_1, g_2]], \quad (12)$$

which is the cocycle condition that $\mathcal{A}$ represents a class in the BRST cohomology of the gauge group. The Becchi–Rouet–Stora construction ([Becchi et al., 1976](#)) formalizes this cocycle structure through the BRST nilpotent operator s . The crucial structural fact is that the anomaly is now classified by a topological invariant of the gauge bundle, completely independent of which quantum theory produced it. Two different chiral fermion contents that give the same characteristic-class data give the same anomaly, because the anomaly is the characteristic class, not the loop calculation. This is the formal-mathematical statement of the ABJ observation that the right-hand side of (11) is a

classical bundle invariant.

The Stora–Zumino descent (Stora, 1984, Wess and Zumino, 1971) then constructs the four-dimensional anomaly $\omega_4^{(1)}$ as the descendant of a higher-dimensional characteristic class through the chain

$$\omega_{2n+2} \rightarrow \omega_{2n+1}^{(0)} \rightarrow \omega_{2n}^{(1)} \rightarrow \omega_{2n-1}^{(2)} \rightarrow \cdots, \quad (13)$$

where ω_{2n+2} is a closed gauge-invariant form on the extended manifold (built from $\text{tr}(R \wedge R)$, $\text{tr}(F \wedge F)$, etc.) and the descendants $\omega^{(k)}$ are obtained by stripping off one exterior power and one BRST ghost at each step. The four-dimensional anomaly is therefore the lowest descendant of a topological characteristic class living naturally on a higher-dimensional bulk. This is the structural identification that the Green–Schwarz mechanism would exploit in 1984.

2.5 Green–Schwarz (1984): the bulk-side rewrite

In ten-dimensional heterotic string theory, Green and Schwarz (1984) discovered that the anomaly polynomial I_{12} of the chiral fermion content for the gauge groups $\text{SO}(32)$ and $E_8 \times E_8$ admits the rank-one factorization

$$I_{12} = X_4 \wedge X_8, \quad (14)$$

where $X_4 = \text{tr}(R \wedge R) - v \text{tr}(F \wedge F)$ is a closed gauge-invariant four-form built from gauge and gravitational curvatures, and X_8 is a closed eight-form. When this factorization holds, GS showed that the boundary chiral anomaly can be *absorbed* by modifying the Bianchi identity of the antisymmetric tensor field strength H_3 :

$$dH_3 = X_4 \quad (\text{in place of } dH_3 = 0). \quad (15)$$

The classical Chern–Simons inflow from this modified H_3 exactly cancels the BRST cocycle of the boundary anomaly.

The deeper content of this move, central to the present paper, is that the modification (15) is a *classical, geometric, topological* statement. The Bianchi identity is being modified to a different cohomology class for H_3 : instead of H_3 being a closed differential form, H_3 is now a representative of a differential character whose curvature class is $[X_4]$. This is a statement about the differential cohomology of the bundle, not about any quantum dynamics. In particular, no quantum-mechanical loop appears in the *statement* of the GS modification; the quantum content enters only via the original ABJ calculation that first revealed the obstruction. The 1984 GS discovery is therefore the geometric rewriting of a classical topological identity, not the quantum cancellation of a quantum effect. As we noted in the historical-positioning subsection of GG-A4-Euclid (Arisaka, 2026, A4-Euclid), the 1984 GS mechanism was the moment higher-dimensional gauge theory glimpsed that gauge curvature F and gravitational curvature R must lie inside a single closed topological four-form on the bulk manifold — a fact that the GG-Constitution later promotes to the axiom AX-2 (GGEP).

2.6 Hopkins–Singer–Freed–Moore: modern topological classification (2000s+)

The modern mathematical formalization of anomalies, developed in the 2000s and 2010s by Hopkins and Singer (2005), Dai and Freed (1994), Monnier (2018), Monnier and Moore (2019), Freed and Hopkins (2021), and others, classifies anomalies through *reflection-positive invertible field theories* on the bordism cohomology of the bundle data. The anomaly of an n -dimensional quantum field theory is encoded by an $(n+1)$ -dimensional invertible TQFT, which is itself a representative of a class in the Anderson dual of an appropriate cobordism spectrum. All of this is *classical* mathematics in the sense that no quantum loops appear in the classification: the anomaly class is a topological invariant of the bundle and the bordism class of the spacetime, and the quantum field theory only *realizes* the invariant by computing a particular amplitude that is sensitive to the underlying topology.

The differential-cohomology refinement of Hopkins and Singer (2005) sharpens this further: characters of $H^*(M; \mathbb{Z})$ are refined to differential characters carrying both topological data (integer class) and geometric data (curvature representative two-form). In this language, the GS modification (15) is the statement that the antisymmetric tensor field is a differential character whose underlying topological class is $[X_4]$, with the connection representing that character supplied by the bulk dynamics. The quantum theory enters only through the choice of which underlying characteristic-class data the bulk realizes; the anomaly itself is fixed by the bundle topology.

This is the formal mathematical confirmation of the reading we are proposing: the anomaly is geometric and topological in its fundamental content, and quantum mechanics is merely the window through which the underlying topology becomes visible at the boundary.

2.7 The cohort constitutional reading: AX-2 on the bulk, AX-3 on the projection

Under the constitutional clauses of GG-A1-Constitution (Arisaka, 2026, A1-Constitution) and the bulk theorem of GG-A2-GG6 (Arisaka, 2026, A2-GG6), the geometric reading developed above can be promoted from a methodological observation to a structural statement. The four axioms of the GG-Constitution are arranged in a strictly nested tower; for the present discussion only the first three are needed.

- **AX-1 (LC, Local Causality):** the bulk manifold admits a smooth causal structure with finite signal speed. This is the prerequisite for any classical or quantum theory to be defined on the bulk.
- **AX-2 (GGEP, Geometry–Gauge Equivalence Principle):** the bulk geometry and the bulk gauge fields form a single composite connection on the 6D manifold $\mathcal{M}_6 = \mathcal{M}_4 \times I_r \times S_\theta^1$. Gauge curvature F and gravitational curvature R are not independent fundamental objects but algebraic facets of one underlying composite connection one-form on the G_{tot} bundle.
- **AX-3 (OGGEP, Operational/Open Geometry–Gauge Equivalence Principle):** any admissible bulk theory must furnish a structure-preserving projection $\mathcal{D}_{\text{OGGEP}} : \mathbf{Bulk} \rightarrow \mathbf{Op}$ from inaccessible bulk data to the registered four-dimensional boundary observables, with the projection compatible with gauge covariance, causality, positivity, and scale consistency.

Concretely, AX-3 says the 4D physics observed by a finite experimenter is the projection of the bulk content under $\mathcal{D}_{\text{OGGEP}}$.

In this constitutional language, the modified Bianchi identity $dH_3 = X_4$ of the present paper is forced on the bulk side by AX-2: the composite connection on the G_{tot} bundle has a closed gauge-invariant four-form X_4 built from the four component curvatures, and the Chern–Simons primitive $H_3 = dB_2 + (\text{Chern-Simons forms})$ then satisfies

$$\boxed{dH_3 = X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - \frac{2}{5} F_B^2 - \frac{2}{5} F_{Qi}^2}, \quad (16)$$

which is the four-component modified Bianchi identity BI-6 of the cohort framework, derived in detail in GG-A2-GG6 and proved anomaly-free in Parts II and III of the present paper. This is a *classical, topological, pre-quantum identity*. It is forced on the bulk by AX-2 alone; AX-3 is not yet invoked.

The four-dimensional boundary observer’s perception of the identity (16) is the OGGEP-projected shadow. Under the projection $\mathcal{D}_{\text{OGGEP}}$, the closed gauge-invariant four-form X_4 descends to a boundary four-form that the boundary observer’s chiral fermion sees as the right-hand side of the divergence equation

$$\partial_\mu J_{\text{bdy}}^\mu \propto \mathcal{D}_{\text{OGGEP}}(X_4), \quad (17)$$

which is the constitutional re-derivation of the ABJ relation (11) on the boundary. The boundary observer registers the divergence as a “quantum anomaly” because she is observing the projected shadow of a classical bulk identity; the appearance of the divergence is forced by AX-3 (it must appear, because the projection is causal, positive, and gauge-covariant), and the specific four-form on the right-hand side is forced by AX-2 (the bundle has $dH_3 = X_4$ on the bulk).

In one line: *the bulk-side identity is the GGEP/AX-2 content; the boundary-side ‘anomaly’ is the OGGEP/AX-3 projected manifestation of the same identity. Constitutionally, they are the same topological fact viewed from opposite sides of the projection.*

The reason the 1984 GS mechanism ‘cancels’ the four-dimensional anomaly is now constitutionally transparent: GS’s modified Bianchi identity is the constitutional statement of what the bulk must be doing in order for the boundary projection to be self-consistent. GS did not cancel a quantum-mechanical effect; GS identified the bulk-side identity whose OGGEP shadow the boundary chiral fermion was already detecting. The customary description of the GS mechanism as a ‘quantum anomaly cancellation’ is accurate as far as it goes, and the reading offered here does not contradict it; it re-describes the same mathematics from the bulk side. On that reading the GS mechanism is the discovery of a classical topological identity that AX-2 forces and that the OGGEP projection makes self-consistent at the boundary. Whether the bulk-side or the boundary-side description is the more natural one is a question of framing, not of fact, and nothing in the theorem below depends on the answer.

2.8 Why this matters for the BI-6 anomaly theorem

The constitutional reading developed in the preceding subsections is the foundational reading on which the BI-6 anomaly theorem of the present paper rests. Three consequences are immediate.

First, the proof object is correctly identified. The object whose anomaly freedom we prove in Parts II and III is BI-6 in the form (16) — the four-component modified Bianchi identity on the 6D bulk. This is a classical topological identity on the bulk, and the consistency theorem we prove is the classical-topological consistency of the bundle plus the global-completion compatibility of the differential-cohomology lift. The boundary observer’s “anomaly” is the shadow, not the object.

Second, the structural method is correctly motivated. The descent proof of Section 5 works with the BRST cocycle structure of the boundary chiral-fermion anomaly precisely because the BRST cocycle is the boundary shadow of the bulk topological identity. Each step in the descent corresponds, in the constitutional reading, to a step in the OGGEF projection from bulk to boundary. The reason the descent argument is the natural proof technique is that it exactly tracks the constitutional projection $\mathcal{D}_{\text{OGGEF}}$ applied to the bulk four-form X_4 .

Third, the global completion is correctly understood. The Wu–Chern–Simons / Hopkins–Singer / Monnier–Moore global proof of Section 6 verifies that the differential character carrying $[X_4]$ is consistently defined on the entire intended GG-6 background class, including under large gauge transformations and across all admissible bulk topologies. This is, in the constitutional reading, the verification that AX-2 forces a globally consistent bulk identity (not merely a locally consistent one), which is the prerequisite for AX-3 to deliver a globally consistent boundary projection.

Figure 1 draws the reading of this section in one picture: the four curvature contributions assemble into the bulk four-form $X_4 = dH_3$ — a classical-topological identity on the composite connection — and the boundary observer’s “anomaly” is its OGGEF-projected shadow.

After all, what the present paper proves is not the cancellation of a quantum-mechanical defect but the constitutional consistency of a classical-topological identity that AX-2 forces on the bulk and that AX-3 projects to the boundary. That this classical consistency theorem coincides with the quantum “anomaly cancellation” theorem of the conventional reading is not a coincidence: the quantum anomaly is what the boundary observer sees when the bulk identity holds. The present paper’s contribution is to prove the bulk identity’s classical consistency; the rest of the cohort program then takes the projected boundary content seriously as the substance of four-dimensional physics.

3 The Green–Schwarz mechanism in even dimensions

3.1 Anomaly polynomials and the descent formalism

Historical background (with citations). Indeed, the modern anomaly polynomial is the formal endpoint of a chain of ideas that begins with classical electrodynamics. Maxwell’s 1865 paper *A Dynamical Theory of the Electromagnetic Field* (Maxwell, 1865) put the four-vector potential A_μ at the center of physics, defined only up to a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \chi$ — a structural fact that, a century later, became the founding example of every gauge-invariant quantum field theory.

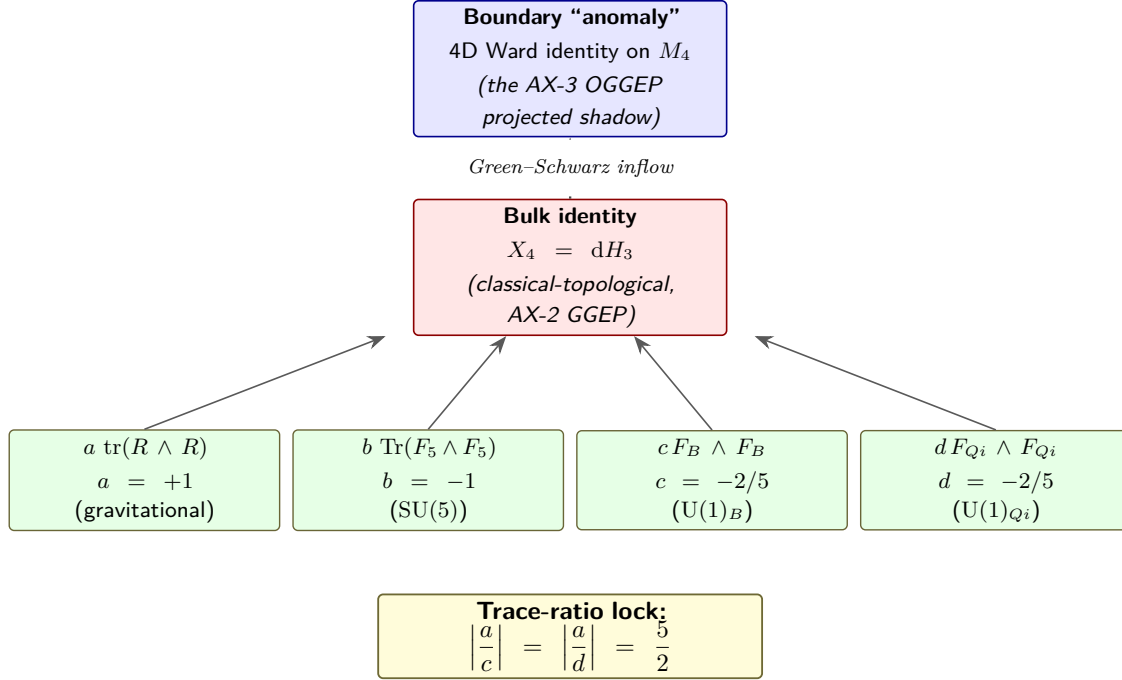


Figure 1: The four-component BI-6 bulk-side classical-topological identity and its boundary shadow. The four curvature contributions (green) carry the canonical coefficients $(a, b, c, d) = (1, -1, -2/5, -2/5)$ and assemble into the bulk four-form $X_4 = dH_3$ (red), a classical identity on the AX-2 GGEF composite connection, pinned by the trace-ratio lock $|a/c| = |a/d| = 5/2$ (yellow; Section 5.1). The boundary “anomaly” (blue) is the AX-3 OGGEF projection shadow of this bulk identity — a boundary re-measurement of a classical fact already in force in the bulk, not a quantum pathology to be cured. Adapted from the cohort master figure of (Arisaka, 2026, A1-Constitution) (its bulk-side identity plate), also carried in (Arisaka, 2026, A2-GG6).

The decisive quantum surprise came in 1969, when Adler (Adler, 1969) and, independently, Bell and Jackiw (Bell and Jackiw, 1969) showed that a chiral fermion coupled to such a gauge field generically violates current conservation at one loop — the *Adler–Bell–Jackiw anomaly*, with $\partial^\mu J_\mu^5 \propto F \wedge \tilde{F}$. This was the moment when “classical symmetry” and “quantum symmetry” became genuinely distinct concepts in physics, and the anomaly polynomial machinery of the present subsection is the systematic descendant of that discovery. The descent equations (18) below are the bookkeeping device that turns the Adler–Bell–Jackiw obstruction into a precise cohomological invariant of the quantum effective action — one whose vanishing or absorption is exactly the “anomaly-freedom” condition we are after.

In a chiral gauge theory in $d = 2n$ dimensions, the local perturbative anomaly is encoded by a closed, gauge-invariant $(2n + 2)$ -form I_{2n+2} , the **anomaly polynomial**. The connection between I_{2n+2} and the anomalous variation of the quantum effective action is established by the **descent formalism**:

$$I_{2n+2} = dI_{2n+1}^{(0)}, \quad \delta I_{2n+1}^{(0)} + dI_{2n}^{(1)} = 0, \quad (18)$$

where δ denotes a gauge or local Lorentz variation. The consistent anomaly of the effective action is

then

$$\delta\Gamma_{1\text{-loop}} = \kappa \int_{M_{2n}} I_{2n}^{(1)}, \quad (19)$$

with κ a normalization constant.

Definition 3.1 (Anomaly freedom). A theory is **perturbatively locally anomaly-free** if $\delta\Gamma_{1\text{-loop}} = 0$ for all infinitesimal gauge and local Lorentz variations. It is **globally anomaly-free** if, in addition, the path integral is single-valued under all large gauge transformations and large diffeomorphisms.

3.2 The Green–Schwarz factorization mechanism

The Green–Schwarz mechanism (Green and Schwarz, 1984) cancels the anomaly when the polynomial I_{2n+2} *factorizes* through a lower-degree form that can be absorbed by a tensor field.

Theorem 3.2 (T-BI6-3.2: Green–Schwarz cancellation, general form). *Let I_{2n+2} factorize as*

$$I_{2n+2} = X_{p+2} \wedge Y_{2n-p}, \quad (20)$$

where X_{p+2} is a closed gauge-invariant $(p+2)$ -form. Introduce a p -form tensor B_p with modified field strength

$$H_{p+1} = dB_p + \omega_p^{(0)}, \quad d\omega_p^{(0)} = X_{p+2}, \quad (21)$$

and transformation law

$$\delta B_p = -\omega_{p-1}^{(1)}, \quad \delta\omega_p^{(0)} + d\omega_{p-1}^{(1)} = 0. \quad (22)$$

Then the Green–Schwarz counterterm

$$S_{\text{GS}} = \kappa \int_{M_{2n}} B_p \wedge Y_{2n-p} \quad (23)$$

cancel the full perturbative local anomaly:

$$\delta\Gamma_{1\text{-loop}} + \delta S_{\text{GS}} = 0. \quad (24)$$

Proof. Since $\delta Y_{2n-p} = 0$ (gauge invariance of characteristic forms),

$$\delta S_{\text{GS}} = \kappa \int \delta B_p \wedge Y_{2n-p} = -\kappa \int \omega_{p-1}^{(1)} \wedge Y_{2n-p}. \quad (25)$$

The one-loop variation, by descent from $I_{2n+2} = X_{p+2} \wedge Y_{2n-p}$, is

$$\delta\Gamma_{1\text{-loop}} = +\kappa \int \omega_{p-1}^{(1)} \wedge Y_{2n-p}. \quad (26)$$

The sum vanishes. ■

3.3 Specialization to six and ten dimensions

In $d = 10$, the anomaly polynomial is a 12-form. The factorization

$$I_{12} = X_4 \wedge X_8 \tag{27}$$

uses a four-form X_4 and an eight-form X_8 , with a two-form tensor B_2 . This is the original Green–Schwarz mechanism.

In $d = 6$, the anomaly polynomial is an eight-form. The single-tensor factorization becomes

$$I_8 = \frac{1}{2}X_4 \wedge X_4 = \frac{1}{2}X_4^2, \tag{28}$$

a **perfect square**. This is more rigid than the 10D case: the same four-form appears twice, so the anomaly polynomial must be a rank-one quadratic form in the space of invariant four-forms.

Remark 3.3. In the 10D case, X_4 and X_8 can be different forms, giving more algebraic freedom. In the 6D case, the perfect-square factorization imposes the rank-one condition on the anomaly matrix, which is a *stronger* constraint. This additional rigidity is the mathematical reason why **BI-6** is so tightly locked.

With the general mechanism and the specific BI-6 data in hand, the proof itself awaits.

4 The cohort inheritance and the roadmap of the ten Parts

Before the anomaly proof begins (Part II), it is useful to lay out two reader-orientation maps on the same opening: a panorama of the twenty-two-paper cohort in which this paper stands (§4.1, Figure 2), and a panorama of the ten Parts of the present paper (§4.2, Figure 3). A reader of any later Part should be able to flip back to this Section and orient themselves in both directions: which cohort companion carries which imported result, and where the present argument stands in the ten-Part arc.

4.1 The cohort inheritance

The present paper is one of twenty-two. **GG-Theory** is written not as a single monograph but as a cohort of twenty-two preprints — eleven A-papers, six P-papers, three C-papers and two white papers — each carrying one rigorously stated piece of a single descent that runs from four axioms and seven postulates to the empirical structure of physics. The partition into twenty-two papers reflects the partition of subject matter, not a ranking of importance: the framework is one program, and each paper is a named carrier of named content. Figure 2 is the panorama; a reader meeting the program for the first time can read its architecture off the figure top to bottom before returning to the anomaly proof that is the business of this paper. The cohort falls into five strata plus three framed branches.

The constitutional root and the GG6 Trio. At the root stands GG-A1-Constitution ([Arisaka, 2026, A1-Constitution](#)), the GG-Constitution: the four axioms (AX-1 LC, AX-2 GGEP, AX-3 OGEP,

AX-4 MSMF) and seven postulates against which every companion’s theorems are audited. Forced from it is the GG6 Trio that builds the bulk: GG-A2-GG6 (Arisaka, 2026, A2-GG6) constructs the six-dimensional Geometry-Gauge bulk and proves it the unique anomaly-free, moduli-free solution of the axioms; the present paper, GG-A3-BI6 (Arisaka, 2026, A3-BI6), supplies the rigorous local-and-global proof that the modified Bianchi identity BI-6 on that bulk is anomaly-free; and GG-A4-Euclid (Arisaka, 2026, A4-Euclid) traces the historical lineage from Euclid through Riemann, Maxwell, and Einstein to the bulk and reconstructs the universal constants from the bulk dimensions.

The boundary law and its two pillars. AX-3 (OGGEP) forces the open-system boundary law GG-A5-GDOS (Arisaka, 2026, A5-GDOS), the Geometric Dynamics of Open Systems that projects bulk physics onto the four-dimensional boundary and recovers the textbook quantum-mechanical postulates as theorems of the operational dictionary. Two sibling pillar papers organize the boundary content: GG-A6-GRIP (Arisaka, 2026, A6-GRIP), the geometric Gi-4 pillars (the cascade from the total geometric-gauge object through the free-energy functional to the gauge-runtime pipeline), and GG-A7-Qi4 (Arisaka, 2026, A7-Qi4), the quantum-information Qi-4 pillars, which carry the Arisaka equation and the boundary $U(1)_{Qi}$ symmetry that the present paper admits.

The Consequence Trio and the capstone. Three papers read the empirical world off the boundary law: GG-A8-QM (Arisaka, 2026, A8-QM) recasts four-dimensional quantum mechanics, GG-A9-SM (Arisaka, 2026, A9-SM) derives the twenty-eight free parameters of the Standard Model, and GG-A10-Cosmos (Arisaka, 2026, A10-Cosmos) derives the Λ CDM cosmological inventory ($\Omega_m = 1/\pi$ and its partners). The capstone GG-A11-Origin (Arisaka, 2026, A11-Origin) closes the cohort with the recursive-uniqueness argument — why every electron is the same electron — and the cosmos–life–mind chain.

The two phenomenology branches. Beneath the A-series, two framed branches carry the sharp falsifiers. The Predictions Branch — GG-P1-KK (Arisaka, 2026, P1-KK) (the Kaluza–Klein lock cluster near $M_{\text{lock}} \approx 3.85$ TeV), GG-P2-N3 (Arisaka, 2026, P2-N3) (the three-generation pincer that forces $N_g = 3$), and GG-P3-Electron (Arisaka, 2026, P3-Electron) (the electron as the ground state of nature), and GG-P6-Proton (Arisaka, 2026, P6-Proton) (the proton as the ground geometric attractor of matter, its decay forbidden by the geometry rather than merely unobserved) — supplies collider and structural predictions. The Cosmology and Astrophysics Branch — GG-C1-Kaxion (Arisaka, 2026, C1-Kaxion) (the kaxion dark-matter candidate near $8.1 \mu\text{eV}$), GG-C2-Hubble (Arisaka, 2026, C2-Hubble) (the Hubble tension as one geometric corridor), and GG-C3-BH (Arisaka, 2026, C3-BH) (black-hole entropy as an OGGEP port) — supplies the astrophysical signatures.

The Foundations Branch. The third framed branch is the Foundations Branch: the two member papers whose subject is the boundary’s complex structure and its mirrors. GG-P4-Real (Arisaka, 2026, P4-Real) stands on the map’s left flank as the origin-of- i member, deriving the single imaginary unit of boundary physics from the two real structures of the GG-A2-GG6 arena — the compact phase circle and the causal clock. GG-P5-CPT (Arisaka, 2026, P5-CPT) carries the three mirrors of matter — C, P, and T read as orientation reversals of the three real bulk factors — and that

orientation reading of the discrete symmetries gives the anomaly-inflow logic of the present paper an independent consumer among the companions. Both carry P-series designators; their place at the spine’s flank, not the phenomenology rows, marks their foundational role.

The White-Paper Branch. The right flank mirrors the left and does the opposite job: where the Foundations Branch feeds the spine, the White-Paper Branch reads it. GG-W1-Symmetry (Arisaka, 2026, W1-Symmetry) is the diachronic one — it walks the century of symmetry and conservation in the order it was found and shows the conviction that symmetry causes conservation loosening at seven historical steps, until what remains conserved is an integer with nowhere to move. GG-W2-Holonomy (Arisaka, 2026, W2-Holonomy) is the synchronic one — it takes one exact statement, that a holonomy requires a closed loop and the radial interval has none, so the circle remembers and the interval forgets, and follows it to sixteen questions physics has not closed. Neither derives anything new: both carry every result at its parent’s declared grade, which is what makes them white papers rather than a third phenomenology branch.

Where this paper sits. GG-A3-BI6 is the second descent step of the cohort. GG-A2-GG6 forces the bulk and states BI-6; the present paper proves that the resulting theory is quantum-mechanically consistent, and the anomaly ledger behind GG-A2-GG6’s Theorem 6 — the OGN-5 integer cascade — is the ledger proved in full here. Every downstream paper — the boundary law of GG-A5-GDOS, the pillars of GG-A6-GRIP and GG-A7-Qi4, the textbook readouts of GG-A8-QM–GG-A10-Cosmos, and the capstone GG-A11-Origin — inherits its license to build on the GG-6 bulk from the anomaly theorem proved here. Without this proof, the third step (GDOS lawfulness) cannot legitimately be invoked. The twenty-two-paper map, identical in every paper of the cohort, is Figure 2, which closes this subsection.

4.2 Roadmap of the ten Parts

The rest of the paper is organized into nine further Parts, and the walk through them is plotted in Figure 3. The present Part — the introduction — is nearly done: the anomaly problem and its constitutional frame, the hundred-and-twenty-year geometric arc from Bianchi to differential cohomology, the Green–Schwarz mechanism in general even dimensions that fixes the notation and the cohomological framework, and the cohort the paper stands in. Everything after this page is the proof and its consequences, and the itinerary below says where each piece lives.

Part II is the heart, and it is highlighted in the roadmap because it is what the title claims. The local descent argument closes one-loop consistency through BRST exactness, descending from the factorized polynomial $I_8 = \frac{1}{2}X_4^2$; the global completion verifies the four hypotheses (H1)–(H4) of the Wu–Chern–Simons / Hopkins–Singer / Monnier–Moore differential-cohomology framework on the intended GG-6 background class; and an explicit benchmark spectrum proves that the structural theorem is satisfiable rather than vacuous. This Part carries the single result on which the rest of the paper — and the cohort’s license to build on the GG-6 bulk — depends.

Two Parts then work the consequences outward. Part III shows that the five-integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ is a theorem-level consequence of BI-6 rather than an external input:

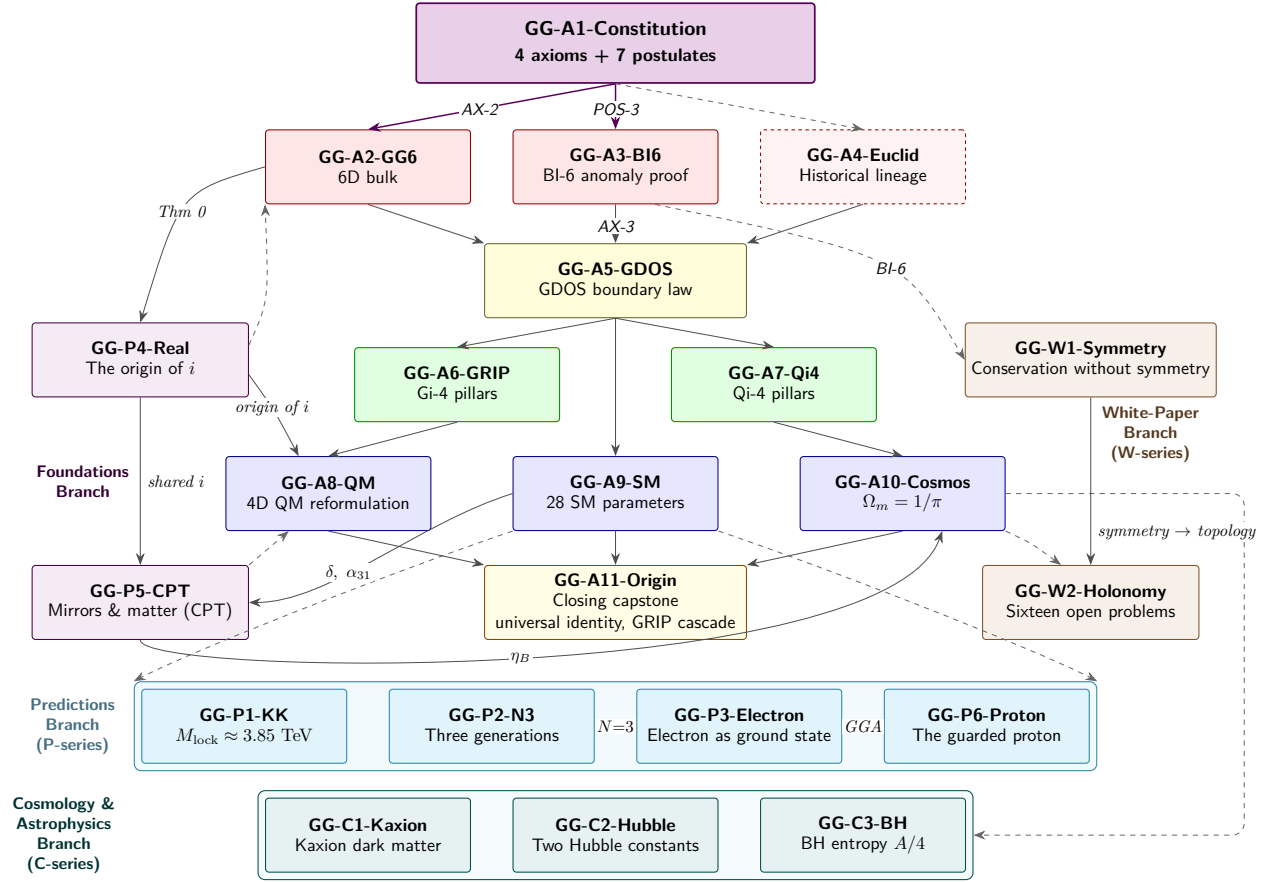


Figure 2: The twenty-two-paper cohort map of GG-Theory. The constitutional root GG-A1-Constitution (violet, top) supplies the four axioms and seven postulates from which the rest of the cohort descends: the GG6 Trio (GG-A2-GG6, GG-A3-BI6, GG-A4-Euclid, red) realizes the bulk geometry; GG-A5-GDOS (yellow) the boundary law; the GDOS Trio pillars (GG-A6-GRIP, GG-A7-Qi4, green) the geometric and quantum-information runtime; the 4D-Physics Trio (GG-A8-QM, GG-A9-SM, GG-A10-Cosmos, blue) the textbook readouts; and the closing capstone GG-A11-Origin (gold) the quarks-to-consciousness identity cascade. Two framed branches carry the phenomenology — the **Predictions Branch** (cyan), whose four papers hand their results along the row by the internal arrows (GG-P2-N3’s $N=3$ into GG-P3-Electron, and that paper’s ground-attractor method, GGA, on into GG-P6-Proton), and the **Cosmology & Astrophysics Branch** (teal). The spine is flanked by two columns of two, which do opposite jobs. On the left the **Foundations Branch** (violet) feeds it: GG-P4-Real, fed by GG-A2-GG6’s Theorem 0 arena, manufactures the i that GG-A8-QM consumes, and GG-P5-CPT is the GG-A9-SM–GG-A10-Cosmos interface, fed by the ledger phases and feeding the η_B chain across the width of the map, with the shared- i edge descending solid between them. On the right the **White-Paper Branch** (brown) reads it: GG-W1-Symmetry walks the century from Noether to Hopkins–Singer, and GG-W2-Holonomy dissects the structure that arc arrives at — protection by topology rather than by symmetry — across sixteen open problems. The white papers derive nothing new, carrying every result at its parent’s declared grade; the single dashed feed drawn into each names its heaviest source, not its only one. *Reading conventions:* solid arrows mark axiom-forced descent and carry an edge label naming the driving axiom or postulate where most illustrative; dashed arrows mark the deployment of derived results, or historical links inherited without being constitutionally forced.

a Diophantine cascade in which each integer is the unique solution of an arithmetic condition fixed by a prior theorem. Part IV sets the construction beside the string program it generalizes — the ten-dimensional Green–Schwarz mechanism, six-dimensional supergravity and F-theory, the AdS/CFT correspondence, and the Randall–Sundrum model — acknowledging the shared mathematics in full, stating the structural departures feature by feature, and naming the open bridges to the string community.

Two Parts then weigh the case. Part V is the discussion: the significance of the result, what interlocks with what to make it unique, the ten structural innovations that are new here, the predictions that can falsify it, the cross-paper coherence map, and the honest open-problems triage with the future directions it sets. Part VI states the conclusions — what has been established, first in plain language for a broad readership and then in the precise register of the theorem chain — and closes with the place of the result in the series.

Part VII then steps out of the technical register entirely and answers, in plain language and assuming no physics beyond a first course, the twenty-five questions a reader without a physics degree actually carries into a paper like this one. It is placed here, before the back matter, because a reader who wants those answers should not have to cross sixty pages of bordism theory to reach them.

The back matter is the workshop. Part VIII collects the technical appendices: a graduate-level course in ten appendices, from the descent formalism and the ten-dimensional Green–Schwarz mechanism worked from scratch to the OGN-5 derivation, the gravitational completion, the global (Dai–Freed) proof, and the theorem checklist matrix. Part IX answers the fifty-nine objections a particle theorist, string theorist, or differential-cohomology specialist is most likely to raise. Part X closes with the annotated references, each carrying a note on what it established and how it is used here. Read in order, the ten Parts compose into a single self-contained anomaly proof that any reader can audit step by step; read by need, each Part names its imports and can stand a visit on its own.

Three routes through this paper. Not every reader wants the same thing from it, and the paper is built so that three different visits all work.

The general reader, who wants to know what is claimed and how much to believe: the Executive Summary (page 2), then Section 18, which states the result in plain language with no notation, then Part VII, which answers twenty-five questions without assuming a physics degree. About twenty pages, and no mathematics is needed at any point.

The specialist auditing the proof: Table 2 first, which grades every claim in the paper and names its status, then the weakest row of that table. Then Part II for the theorem, Appendices B, D and I for the calculations behind it, and Part IX for the fifty-nine objections, several of which concede something the main text has to be read down for.

The reader who has come from another cohort paper and needs one result: Section 4.1 says what this paper supplies and to whom, Section 9 carries the integer ledger, and Appendix J is the one-row-per-claim ledger with each claim’s status and where it is established.

What the reader now has, and what the proof will cost. Four things have been put in place.

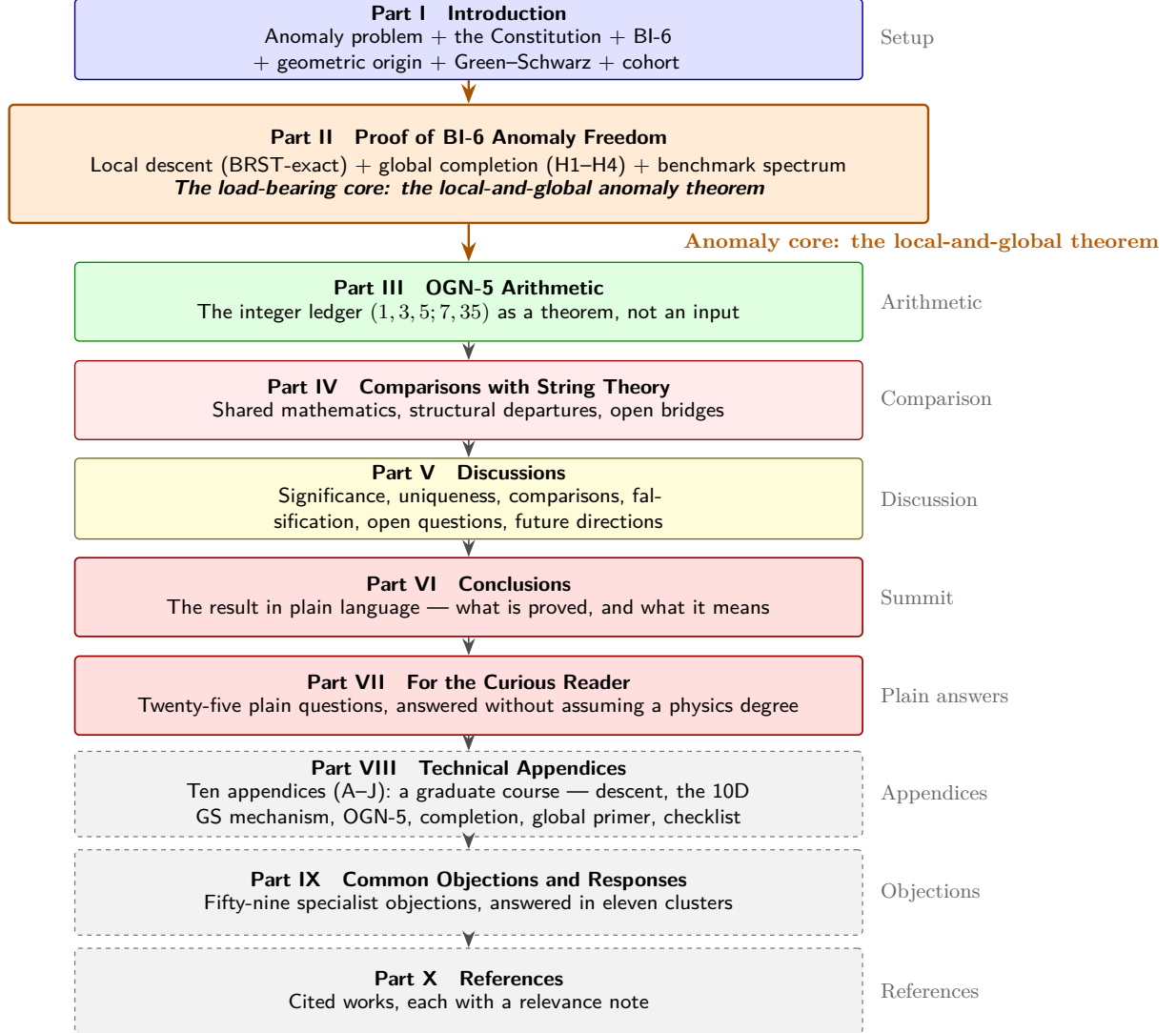


Figure 3: Roadmap of the ten Parts of GG-A3-BI6. The argument develops in phases. *Setup* (Part I) states the anomaly problem, the GG-Theory Constitution and the BI-6 identity it forces, the geometric origin of the anomaly concept, the Green–Schwarz mechanism in general even dimensions, and the cohort inheritance with this roadmap. The *anomaly core* (**Part II**, highlighted) is the technical heart and the title claim: the local descent argument closes one-loop consistency through BRST exactness, the global completion verifies the four hypotheses (H1)–(H4) of the Wu–Chern–Simons / Hopkins–Singer / Monnier–Moore differential-cohomology framework on the intended GG-6 background class, and the benchmark spectrum proves the structural theorem satisfiable. *Arithmetic* (Part III) derives the OGN-5 ledger (1, 3, 5; 7, 35) as a theorem-level consequence of BI-6; *comparison* (Part IV) sets the construction beside the ten-dimensional Green–Schwarz mechanism, six-dimensional supergravity and F-theory, AdS/CFT, and Randall–Sundrum; *discussion* (Part V) weighs significance, uniqueness, falsification, and the open-problems triage; and the *summit* (Part VI) states the conclusions in plain language first and in the precise register of the theorem chain second. *Plain answers* (Part VII) then takes twenty-five questions a reader without a physics degree actually asks, and answers them assuming no physics beyond a first course; it is placed before the back matter because a reader who wants it should not have to cross sixty pages of bordism theory to reach it. The three back-matter Parts collect the technical appendices (Part VIII), the fifty-nine specialist objections and responses (Part IX), and the annotated references (Part X). Part II is highlighted because the local-and-global anomaly-freedom theorem is the single result on which the rest of the paper — and the cohort’s license to build on the GG-6 bulk — depends.

First, a statement of the problem: an anomaly in the gauge symmetry is fatal, not cosmetic, and a six-dimensional chiral theory has to survive it before anything else about that theory means anything. Second, a reading of where the problem comes from — it is geometric, and it was geometric before quantum mechanics existed, which is why a classical identity about curvature can be the thing that resolves it. Third, the general mechanism: if the anomaly polynomial factorizes, one tensor field can absorb it, and the factorization condition is severe enough to be worth checking. Fourth, the object to be tested: the six-dimensional bulk of GG-6, with its four-factor gauge group and the four-component identity BI-6 that the constitution forces on it.

What follows will cost the reader some patience, and it is worth saying in advance where the cost falls. The local half of the proof, Part II's first section, is a calculation: long, but every step is an algebraic identity that can be checked with paper and pencil, and Appendix A does check them. The global half is harder, and it is harder for a reason worth understanding. Local cancellation says the books balance at every point. Global cancellation says they balance after the theory is carried all the way around a loop in the space of its own configurations — and a quantity can return to its starting value at every point along such a loop and still come back multiplied by a phase. Detecting that requires machinery from topology that has no analogue in the local calculation: bordism groups, characteristic classes, an invariant of a differential operator called the eta invariant. Part II states what that machinery delivers and Appendix H supplies a primer for a reader meeting it for the first time.

A reader who wants only the result, and not the audit, can take the local calculation on trust, read the statement of the global theorem, and go on to Part III, where the arithmetic begins to produce numbers that experiments can check.

Part II

Proof of **BI-6** Anomaly Freedom

Local descent, global completion, and the benchmark spectrum

This Part contains the core mathematical content of the paper. Section 5 proves local anomaly cancellation by descent from the factorized polynomial $I_8 = \frac{1}{2}X_4^2$. Section 6 upgrades that local result to a complete global theorem on the physical intended **GG-6** background class, using the modern Wu–Chern–Simons / differential-cohomology framework. Section 7 constructs an explicit finite chiral spectrum that realizes the factorization, proving that the structural theorem is not vacuous. The detailed computations behind each step are provided in Appendices B–I.

This is the Part the title of the paper refers to, and it is worth saying at the outset exactly what has to be established. A complete claim of anomaly freedom is not one statement but eight, and Section 5.4 lists them as proof obligations **(PO1)–(PO8)**. In plain English they are these. The anomaly has to be computed from the actual matter content rather than assumed. The terms that no counterterm can ever cancel — the irreducible ones — have to vanish outright. The mixed terms that would couple one force to another in the wrong way have to vanish too. What remains has to match the square of the **BI-6** four-form exactly, with no coefficient left free to be adjusted. The counterterm’s variation under a gauge transformation has to cancel the leftover, not merely resemble it. The cancellation has to be exact at the level of the cocycle, which is the algebraic form the statement takes when the bookkeeping is done carefully. Global anomalies — the phases that only appear after a loop — have to be absent. And the counterterm itself has to be a well-defined object on the lattice of charges, not just a formal expression.

The eight are not equally hard. Four of them are the calculation. Three are topology. One, the last, is the question of whether the object we have been manipulating exists at all in the strict sense, and it is the one that a reader from the older literature is most likely to underestimate. All eight are discharged in this Part or in the appendices it points to, and Section 5.4 says which result closes which.

5 Structural proof of **BI-6** anomaly cancellation

5.1 The **BI-6** four-form and its coefficient vector

The **GG-6** bulk has gauge-gravity group (4). With a single Green–Schwarz two-form B_2 , the most general diagonal single-tensor candidate four-form is

$$X_4^{\text{gen}} = a \operatorname{tr} R^2 + b \operatorname{Tr} F_5^2 + c F_B^2 + d F_{Q_i}^2. \quad (29)$$

The BI-6 proposal is

$$X_4^{\text{BI6}} = \text{tr } R^2 - \text{Tr } F_5^2 - \frac{2}{5} F_B^2 - \frac{2}{5} F_{Q_i}^2. \quad (30)$$

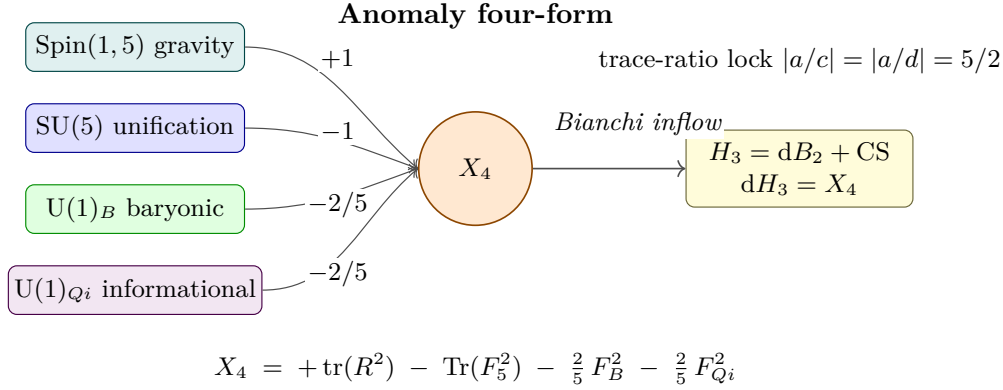


Figure 4: Four-component anomaly inflow in BI-6. The bulk gauge-gravity group $G_{\text{Bulk}} = \text{Spin}(1,5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$ contributes four distinct curvature-squared four-forms which combine, with OGN-locked coefficients $(+1, -1, -2/5, -2/5)$, into a single closed gauge-invariant four-form X_4 . The trace-ratio lock $|a/c| = |a/d| = 5/2$ pins the Abelian fraction at $-2/5$ symmetrically across $\text{U}(1)_B$ and $\text{U}(1)_{Q_i}$ under the equal-Abelian-treatment convention of AX-4. The modified Bianchi identity $dH_3 = X_4$ closes the entire anomaly inflow through the single tensor sector $H_3 = dB_2 + (\text{Chern-Simons forms})$. This is the rank-one single-tensor Green-Schwarz factorization that Theorem 5.1 together with Theorem 5.5 prove to be the unique axiomatic choice.

The coefficient $a = +1$ is fixed by primitive normalization of the tensor field. The sign $b = -1$ is fixed by the Green-Schwarz inflow requirement (opposite signs for Lorentz and gauge Chern-Simons forms). The Abelian coefficient $2/5$ is the trace ratio

$$\frac{\text{Tr}(B_0^2)}{\text{Tr}(Y_0^2)} = \frac{1/3}{5/6} = \frac{2}{5}, \quad (31)$$

where

$$Y_0 = \text{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right), \quad B_0 = \text{diag}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right) \quad (32)$$

are the standard hypercharge and baryon generators in the **5** of $\text{SU}(5)$. The three color entries carry baryon charge $1/3$ and the two lepton entries carry zero, while the hypercharge assignment is $-1/3$ for color and $+1/2$ for leptons (the Georgi-Glashow embedding (Georgi and Glashow, 1974)). The traces are computed directly:

$$\text{Tr}(B_0^2) = 3 \times \left(\frac{1}{3}\right)^2 + 2 \times 0^2 = \frac{1}{3}, \quad \text{Tr}(Y_0^2) = 3 \times \left(\frac{1}{3}\right)^2 + 2 \times \left(\frac{1}{2}\right)^2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}. \quad (33)$$

The ratio is therefore $\text{Tr}(B_0^2)/\text{Tr}(Y_0^2) = (1/3)/(5/6) = 2/5$, which fixes the Abelian coefficient in (30). Two remarks keep the bookkeeping honest. First, B_0 is neither traceless nor central, so it is not an $\text{SU}(5)$ -commuting generator; the number does not depend on treating it as one, because over

a full Standard-Model generation with component-resolved baryon number the same ratio returns exactly, $\sum B^2 / \sum Y^2 = (4/3)/(10/3) = 2/5$ — a generation trace, well defined without any bulk generator. Second, $\sum BY = 2/3 \neq 0$ over a generation, so $U(1)_B$ and hypercharge mix kinetically at one loop; the capacity theorem below is a statement about the ledger share and does not address that mixing. The equality $c = d$ follows from equal Abelian treatment under AX-4/POS-6.

5.2 The geometric and topological reading: BI-6 as a Chern–Weil identity for the B -field gerbe

The four-form (30) and its coefficient vector $(1, -1, -\frac{2}{5}, -\frac{2}{5})$ carry a transparent geometric and topological meaning, which we now develop from first principles. The aim is pedagogical as much as structural: a reader meeting the modified Bianchi identity for the first time — a beginning graduate student as readily as a senior specialist — should come away seeing that $dH_3 = X_4$ is not an *ad hoc* device but the *Chern–Weil consistency condition of a connection*, written one categorical level above the familiar gauge field. We proceed in seven self-contained steps.

Step 1: the ordinary Bianchi identity is a fiber bundle’s Chern–Weil consistency. Let $P \rightarrow \mathcal{M}_6$ be a principal bundle whose structure group is one of the factors of G_{Bulk} , with gauge potential A (a Lie-algebra-valued one-form — the *connection* of Ehresmann (1950)) and curvature

$$F = dA + A \wedge A. \quad (34)$$

The curvature obeys the *Bianchi identity*

$$DF := dF + [A, F] = 0, \quad (35)$$

which is not an extra postulate but an automatic consequence of $d^2 = 0$ applied to (34). Its value is that it makes the *Chern–Weil* construction work: for any invariant polynomial $\mathcal{P}$ of the curvature, (35) gives $d\mathcal{P}(F) = 0$, so $\mathcal{P}(F)$ is closed, and changing the connection $A \rightarrow A'$ shifts it only by an exact form. Hence the de Rham class $[\mathcal{P}(F)]$ is a topological invariant of the bundle — a *characteristic class* — independent of the connection (Milnor and Stasheff, 1974). The two basic four-form cases are

$$\frac{1}{8\pi^2} \text{tr } F^2 \mapsto \text{ch}_2(P) \in H^4(\mathcal{M}_6; \mathbb{Z}), \quad \frac{1}{16\pi^2} \text{tr } R^2 \mapsto \frac{1}{2} p_1(T\mathcal{M}_6) \in H^4(\mathcal{M}_6; \mathbb{Z}), \quad (36)$$

the second Chern character of a gauge bundle and half the first Pontryagin class of the tangent bundle. *Each of the four curvature-squared four-forms in X_4 — $\text{tr } R^2$, $\text{Tr } F_5^2$, F_B^2 , $F_{Q_i}^2$ — is therefore, up to normalization, a representative of an integral degree-four characteristic class.*

Step 2: the Green–Schwarz tensor is a connection one level up — a gerbe. A $U(1)$ gauge field is a connection on a circle bundle; its curvature $F = dA$ is a two-form whose class $[F/2\pi]$ lies in $H^2(\mathcal{M}_6; \mathbb{Z})$ (the first Chern class), and the holonomy $\exp \oint A$ is the observable phase (Wu and Yang, 1975). The two-form potential B_2 of BI-6 is the next object in this sequence: a connection

on a *gerbe* — equivalently a bundle gerbe (Murray, 1996) or a 2-bundle — whose curvature is the three-form $H_3 = dB_2$, with class $[H_3/2\pi] \in H^3(\mathcal{M}_6; \mathbb{Z})$, the *Dixmier–Douady* class (Brylinski, 1993). The dictionary is exact, and Table 3 states it. The Green–Schwarz tensor of BI-6 is thus a gerbe

Table 3: The step from a circle bundle to a gerbe, written as a dictionary. The first row is the familiar case — a gauge field with a two-form curvature and a Chern class — and the second is the object one degree higher, with a two-form potential, a three-form curvature and a Dixmier–Douady class. The correspondence is exact rather than analogical, which is what licenses the construction in this Part: the tensor field of the anomaly mechanism is a connection on a gerbe in precisely the sense that a gauge field is a connection on a bundle.

object	potential	curvature (integral class)
circle bundle	gauge field A	$F = dA$ (H^2 , Chern)
gerbe	tensor B_2	$H_3 = dB_2$ (H^3 , Dixmier–Douady)

connection and H_3 is its field strength: the bulk anomaly inflow of Figure 4 lives “one categorical level above” the gauge sector.

Step 3: Chern–Simons transgression forces the modified Bianchi. A closed invariant four-form is locally exact: each curvature-squared term is the differential of a *Chern–Simons* three-form (Chern and Simons, 1974),

$$\mathrm{tr} F^2 = d\omega_3(A), \quad \omega_3(A) = \mathrm{tr} \left(A dA + \frac{2}{3} A^3 \right), \quad (37)$$

and likewise $\mathrm{tr} R^2 = d\omega_3(\Omega)$ for the Lorentz connection Ω . The form ω_3 is *not* globally well-defined — it shifts under large gauge transformations — which is precisely why $[\mathrm{tr} F^2]$ can be a non-trivial class. The Green–Schwarz prescription absorbs these three-forms into the gerbe connection,

$$H_3 = dB_2 + a\omega_3(\Omega) + b\omega_3(A_5) + c\omega_3(A_B) + d\omega_3(A_{Qi}), \quad (38)$$

so that, taking d and using (37) with the coefficients of (30),

$$dH_3 = a \mathrm{tr} R^2 + b \mathrm{Tr} F_5^2 + c F_B^2 + d F_{Qi}^2 = X_4. \quad (39)$$

The modified Bianchi identity $dH_3 = X_4$ is thus *forced*: it is the field strength of the Chern–Simons-corrected gerbe connection (38). The naive $dH_3 = 0$ is the special case of a gerbe with no characteristic-class source.

Step 4: the coefficient vector is a characteristic-class matching condition. Because $X_4 = dH_3$ is *exact*, its de Rham class vanishes:

$$[X_4] = a [\mathrm{tr} R^2] + b [\mathrm{Tr} F_5^2] + c [F_B^2] + d [F_{Qi}^2] = 0 \quad \text{in } H^4(\mathcal{M}_6; \mathbb{R}). \quad (40)$$

Equation (40) is a condition on the *background*, not an identity of the coefficient vector: it is the tensor tadpole condition, and backgrounds on which the weighted class $[X_4]$ fails to vanish (a spin base with $p_1 \neq 0$ and trivial gauge bundles, say) admit no globally defined H_3 and are excluded from the admissible class by the requirement that the differential cocycle $\check{B}_2$ exist. Read this way, the vector $(a, b, c, d) = (1, -1, -\frac{2}{5}, -\frac{2}{5})$ is the combination the ledger fixes, and (40) is the admissibility condition its backgrounds must satisfy — the four degree-four characteristic classes — gravitational, $SU(5)$, and the two Abelian — *cancel in cohomology* on the admissible GG-6 background. The “trace-ratio lock” $|a/c| = |a/d| = 5/2$ of (31) is this matching condition expressed through the Georgi–Glashow embedding. Equivalently, a globally defined H_3 exists *iff* $[X_4] = 0$: condition (40) *is* the anomaly-cancellation requirement, now visibly a statement about the topology of the gauge and tangent bundles rather than about Feynman diagrams. (This matching condition is the existence condition for a globally defined H_3 — the tensor tadpole; it is logically distinct from, and complementary to, the factorization condition $I_8 = \frac{1}{2}X_4^2$ on the chiral spectrum enforced in Sections 5.4 and 7.)

Step 5: anomaly inflow as the Stora–Zumino descent. The same three-form organizes the cancellation of the boundary anomaly. From the gauge-invariant X_4 one descends to the gauge-variant Chern–Simons form ω_3^0 (with $X_4 = d\omega_3^0$) and then to the anomaly two-form ω_2^1 (with $\delta\omega_3^0 = d\omega_2^1$, δ the gauge variation) — the descent triangle reviewed in §2. The Green–Schwarz counterterm $\int B_2 \wedge X_4$ then has a gauge variation that cancels the one-loop anomaly of the bulk spectrum: the bulk H_3 -flux, sourced by X_4 through (39), is the anomaly inflow of Callan and Harvey (1985). What the interval’s ends carry is settled by the termination of Section 5.7 — a boundary condition, not this variation — and its level-by-level check is Open Problem O13. In the constitutional reading of this program that inflow is the bulk content of AX-2 (GGEP), and its operational projection to the boundary is AX-3 (OGGEP); the modified Bianchi is therefore a *consistency* condition, not an equation of motion.

Step 6: integrality and the OGN-5 ledger. Characteristic numbers — integrals of characteristic classes over closed four-cycles — are *integers*. This is the same integrality that quantizes magnetic charge as the Chern number of a $U(1)$ bundle (Wu and Yang, 1975) and instanton number as ch_2 . The matching condition (40) is accordingly not an equation among arbitrary reals but among *integer* characteristic numbers weighted by group-theoretic trace ratios. Imposing it together with the rank-one single-tensor requirement confines the coefficient vector to a one-dimensional rational lattice direction and fixes the gravitational and non-Abelian entries. Integrality alone does not fix the Abelian entry along that direction; that entry is fixed separately, by the trace-ratio identity of Theorem 8.1. The integer content of the resulting vector is exactly the OGN-5 ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$, derived in §9. The small integers of GG-6 are, in this light, *not* numerology: they are the integrality of characteristic classes — the most robust topological quantization known.

Step 7: the rigorous home, and an honest caveat. At full rigor the tensor B_2 is a connection on a bundle gerbe (Brylinski, 1993, Murray, 1996), equivalently a cocycle in *differential cohomology*

(Hopkins and Singer, 2005), and the modified Bianchi identity is the curvature equation of that differential cocycle. The integral lift of X_4 and the precise quantization of the H_3 -flux are differential-cohomology and (twisted) K -theory statements; for six-dimensional non-supersymmetric theories of the present type the applicable framework is that of Freed and Hopkins (2021), Hopkins and Singer (2005), Monnier and Moore (2019), whose conditions §5.1 and the global-anomaly analysis verify. “Fiber bundle” is thus the correct *intuition*, while the precise object carrying H_3 is a higher bundle; we keep the bundle language for its clarity and flag this one categorical step. With the caveat noted, BI-6 is exactly the Chern–Weil identity of the Green–Schwarz gerbe, and its coefficient vector is a matching of characteristic classes.

Synthesis: one idea, applied twice. The whole structure is a single idea used at two levels. A gauge field is a connection; its curvature obeys the Bianchi identity; invariant polynomials of the curvature are characteristic classes (Chern–Weil). The Green–Schwarz tensor is a connection one level up; its curvature obeys the *modified* Bianchi identity $dH_3 = X_4$; and X_4 is a characteristic four-form whose vanishing cohomology class is the anomaly-cancellation condition. BI-6 is the name, in this program, of that one equation specialized to the forced six-dimensional bulk, and the coefficient vector $(1, -1, -\frac{2}{5}, -\frac{2}{5})$ is the topology of the Standard-Model embedding written as a matching of characteristic classes. The anomaly is geometry; its cancellation is topology; the integers are characteristic numbers — a chain we hope is as inviting to a student meeting it now as it has been to the geometers and physicists who forged each link.

5.3 Anomaly matrix and the rank-one criterion

Figure 4 (page 41) visualizes the four-component anomaly inflow of BI-6: the four bulk curvatures combine via the OGN-locked coefficient vector $(+1, -1, -2/5, -2/5)$ into a single closed gauge-invariant four-form X_4 , which inflows to the two-form tensor B_2 through the modified Bianchi identity $dH_3 = X_4$.

Introduce the shorthand

$$x = (\text{tr } R^2, \text{Tr } F_5^2, F_B^2, F_{Q_i}^2)^T, \quad v_{\text{BI6}} = (1, -1, -2/5, -2/5)^T. \quad (41)$$

Then

$$X_4^{\text{BI6}} = v_{\text{BI6}} \cdot x, \quad I_8^{\text{BI6}} = \frac{1}{2}(v_{\text{BI6}} \cdot x)^2 = \frac{1}{2} x^T M_{\text{BI6}} x, \quad (42)$$

with

$$M_{\text{BI6}} = v_{\text{BI6}} v_{\text{BI6}}^T = \begin{pmatrix} 1 & -1 & -2/5 & -2/5 \\ -1 & 1 & 2/5 & 2/5 \\ -2/5 & 2/5 & 4/25 & 4/25 \\ -2/5 & 2/5 & 4/25 & 4/25 \end{pmatrix}. \quad (43)$$

Theorem 5.1 (T-BI6-6.1: Rank-one criterion). *A single anti-self-dual Green–Schwarz two-form can cancel the reducible part of a six-dimensional local anomaly **if and only if** the 4×4 anomaly matrix has rank one and is positive semidefinite.*

Proof. A single tensor couples to one four-form $X_4 = v \cdot x$. The cancelable polynomial is $\frac{1}{2}(v \cdot x)^2 =$

$\frac{1}{2}x^T(vv^T)x$. The matrix vv^T has rank one and is positive semidefinite. Conversely, a symmetric matrix that is both rank one and positive semidefinite has a single eigenvalue $\lambda > 0$ and factors as vv^T with $v = \sqrt{\lambda}\hat{v}$. The sign hypothesis is not decorative, but it is conventional until the tensor’s duality is fixed: $-vv^T$ is also rank one, and is canceled by a tensor of the *opposite* chirality. The positive-semidefiniteness clause is meaningful once the anti-self-duality of Proposition 5.10 — forced by the gravitational sector — fixes the convention; with that convention, the sign is settled by computation — see the Corollary below. The restriction to a chiral tensor is what makes the “only if” true: a single *non-chiral* two-form couples to two independent four-forms and cancels a rank-two form of signature $(1, 1)$ — the ten-dimensional $I_{12} = X_4 \wedge X_8$, or a $(1, 0)$ theory with one tensor multiplet — and that case is excluded here by clause (C3), not by this theorem. ■

Corollary 5.2. *The BI-6 matrix (43) has rank one, with eigenvalues $\{58/25, 0, 0, 0\}$, confirming that BI-6 passes the single-tensor factorization test.*

Remark on what the rank-one criterion proves and what it does not. A careful reader may worry that the rank-one criterion is tautological: “of course vv^T has rank one, by construction.” The non-trivial content is one step earlier. The theorem asserts that single-tensor cancellation is possible *if and only if* the 4×4 matrix of reducible coefficients *computed microscopically from the chiral spectrum* has rank one — and the microscopic computation is what could in principle have failed. Indeed, a generic spectrum of chiral fermions in six dimensions produces an anomaly matrix of rank greater than one; the BI-6 spectrum is the specific construction whose microscopic matrix collapses to rank one with eigenvector $v_{\text{BI-6}} = (1, -1, -2/5, -2/5)^T$. The rank-one corollary above therefore is an algebraic identity, not an independent measurement: the matrix is *defined* as an outer product, so its rank is one by construction. It is worth being exact about which part is an output. Rank-one collapse is the output: a generic six-dimensional chiral spectrum does not have it, and the fact that this one does is what makes single-tensor inflow available. The particular *value* of the Abelian entry is not. The benchmark spectrum of Section 7 is constructed by matching to a target polynomial in which that entry already appears, so the eigenvector is unique *given* the spectrum, and the spectrum is what it is because the entry was fixed beforehand. The Abelian entry is fixed by the trace-ratio identity of Theorem 8.1, not by the rank-one condition, and Section 9 says so. Completing the concession: since the matrix is an outer product by construction, the genuine theorem content is *satisfiability* — that a finite chiral spectrum exists whose microscopic anomaly equals $\frac{1}{2}X_4^2$ — and that is what Section 7 proves.

Uniqueness of the rank-one factorization. A second concern is whether the eigenvector $v_{\text{BI-6}}$ is itself unique. It is, up to an overall sign. If $vv^T = ww^T$ as 4×4 symmetric matrices, then writing both sides in their orthogonal eigendecomposition shows that w must lie along the same line as v , and the rank-one condition $w \cdot w^T = vv^T$ together with the fixed gravitational normalization $a = 1$ pins $w = \pm v$. The sign is then fixed by the Green–Schwarz convention that the gravitational and gauge sectors enter X_4 with opposite signs (so that $b = -1$, not $b = +1$). Hence the BI-6 coefficient vector $(1, -1, -2/5, -2/5)$ is the unique rank-one factorization of the microscopically computed anomaly polynomial. No other coefficient vector consistent with the spectrum produces a rank-one

matrix.

What a single tensor cannot do, and what that requires of the matter. Before turning to the cancellation itself, it is worth extracting a necessary condition that the rank-one criterion carries but does not advertise. The criterion says the cancellable part of the eight-form is $\frac{1}{2}(v \cdot x)^2$, a perfect square of a form that is *quadratic* in the curvatures. Squaring a quadratic form is restrictive, and it is restrictive in a way that speaks directly about the matter content.

It can manufacture $\text{Tr } F_5^2 \wedge F_B^2$, because that is a product of two entries of x . It can never manufacture the irreducible quartic $\text{Tr } F_5^4$ — the piece that is genuinely not a product of two quadratic traces — because a product of two second-order objects is by construction reducible. And it can never manufacture $\text{Tr } F_5^3 \wedge F_B$ either, for a simpler reason: X_4 contains no cubic invariant anywhere in it, so there is nothing in the square that could supply the $\text{Tr } F_5^3$.

A generic chiral spectrum in six dimensions produces both of those monomials. For them there is no channel. The bulk is not free to generate them and leave the Green–Schwarz term to absorb them, because no such term exists; they must cancel *among the fermions themselves*, before the inflow mechanism is invoked at all. In the usual model-building idiom one says “choose a spectrum whose anomalies cancel.” The single-tensor architecture says something sharper, and it is a statement about the mechanism rather than about GG-6:

Theorem 5.3 (T-BI6-6.2: Necessary conditions for single-tensor inflow). *Let a six-dimensional chiral theory with simple gauge factor $\text{SU}(L)$ and Abelian factors carry a single Green–Schwarz tensor, so that the entire reducible anomaly is canceled by inflow through one four-form X_4 quadratic in the curvatures. Then the charged content must satisfy*

$$B = \sum_{\rho} \sigma_{\rho} B_{\rho} = 0 \quad \text{and} \quad A_3 = \sum_{\rho} \sigma_{\rho} q_{\rho} A_{3,\rho} = 0, \quad (44)$$

where B_{ρ} is the irreducible quartic index of ρ , $A_{3,\rho}$ its cubic index, q_{ρ} its Abelian charge and σ_{ρ} its six-dimensional chirality. These are conditions on the matter alone: they involve neither the coefficient vector of X_4 nor the tensor normalization, and they must hold before any inflow is computed.

Proof. Expand $\frac{1}{2}X_4 \wedge X_4$, where $X_4 = v \cdot x$ and x is the vector of quadratic invariants

$$x = (\text{tr } R^2, \text{Tr } F_5^2, F_B^2, F_{Q_i}^2).$$

Every monomial produced is a product of two entries of x ; hence the expansion contains no term proportional to the irreducible $\text{Tr } F_5^4$, and none proportional to $\text{Tr } F_5^3 \wedge F_B$ or $\text{Tr } F_5^3 \wedge F_{Q_i}$. Both monomials occur in the microscopic polynomial: for a Weyl fermion in ρ with Abelian charge q , the eight-form part of $\text{ch}(F_5 + qF_B)$ contains $\text{Tr}_{\rho} F_5^4$ and $4q \text{Tr}_{\rho} F_5^3 \wedge F_B$. Since the counterterm cannot reach them, their total coefficients must vanish identically on the spectrum, which is (44). ■

Remark 5.4 (Why this is worth isolating). Theorem 5.3 converts a design choice into an obligation. The core block of Section 7.2 is built so that its irreducible and cubic indices vanish internally; read

through the present theorem, that is not a convenient property of a package we happened to select but a property that *any* admissible content must have. The theorem is also the entry point for Theorem 9.2: carrying L as a symbol in (44) turns the two conditions into two equations in L , and they turn out to be the same equation.

5.4 The eight-part proof obligation

For a complete claim of anomaly freedom, eight conditions must hold. We label them **(PO1)**–**(PO8)** — *proof obligations* — to keep them distinct from the twelve Open Problems O_1 – O_{12} triaged in Section 17 and from the numbered Objections of Part IX.

- (PO1)** The one-loop anomaly polynomial must be computed from the full chiral spectrum.
- (PO2)** Every irreducible term $(\text{tr } R^4, \text{Tr } F_5^4)$ must vanish.
- (PO3)** Every off-diagonal Abelian term $(F_B^3 F_{Qi}, F_B F_{Qi}^3, \text{tr } R^2 F_B F_{Qi}, \text{etc.})$ must vanish.
- (PO4)** The reducible coefficients must match (43) exactly.
- (PO5)** The descent variation of S_{GS} must cancel the one-loop anomaly.
- (PO6)** The anomaly cocycle must be BRST-exact.
- (PO7)** Global gauge anomalies must be absent.
- (PO8)** The exponentiated Green–Schwarz term must be well defined on the charge lattice.

5.5 Microscopic anomaly polynomial

The total anomaly polynomial of the theory is the sum of contributions from every chiral field in the six-dimensional spectrum. Each six-dimensional Weyl fermion of chirality $\sigma_s = \pm 1$ in $\text{SU}(5)$ representation ρ_s with Abelian charges $(q_{B,s}, q_{Qi,s})$ contributes the following eight-form (derived from the index theorem for the Dirac operator on M_6 ; see Alvarez-Gaumé and Witten (1984)):

$$I_8^{(s)} = \sigma_s \left[\frac{d_{\rho_s}}{5760} (7p_1^2 - 4p_2) - \frac{d_{\rho_s}}{48} p_1 x_s^2 - \frac{1}{48} p_1 T_{2,s} + \frac{d_{\rho_s}}{24} x_s^4 + \frac{1}{4} T_{2,s} x_s^2 + \frac{1}{6} T_{3,s} x_s + \frac{1}{24} T_{4,s} \right], \quad (45)$$

where $d_{\rho_s} = \dim(\rho_s)$, $T_{n,s} = \text{tr}_{\rho_s}(f_5^n)$ are the n -th Casimir traces of the $\text{SU}(5)$ field strength, $x_s = q_{B,s} f_B + q_{Qi,s} f_{Qi}$ is the Abelian field-strength combination weighted by the species charges, and p_1, p_2 are the first and second Pontryagin classes. Throughout the anomaly bookkeeping, each curvature-squared slot denotes its characteristic four-form in units of $8\pi^2$, so that $\text{tr } R^2 \mapsto p_1/2$, $\text{Tr } F_5^2 \mapsto c_2(E_5)$, and $F^2 \mapsto c_1(L)^2$ — the normalization of the integral-lift lemma of Appendix I. The first term is the purely gravitational contribution (present even for gauge singlets), the middle terms are mixed gauge–gravity and mixed gauge–Abelian couplings, and the last two terms are the purely non-Abelian cubic and quartic Casimir contributions.

The anomaly-cancellation condition requires that the total polynomial $I_8^{\text{tot}} = \sum_s I_8^{(s)} + I_8^{\text{tensor}} + I_8^{\text{vector}}$ factorize as $\frac{\Omega}{2} X_4^2$. Expanding both sides and comparing term by term yields ten coefficient-matching conditions (for the reducible monomials) and eight vanishing conditions (for the irreducible and forbidden monomials), as listed explicitly in Appendix D.

5.6 The descent proof

The local anomaly cancellation is the central structural theorem of the paper. The proof strategy is standard Green–Schwarz descent, applied to the specific BI-6 factorization. We state the theorem, give the proof with all steps displayed, and then record the corollaries. The fully expanded version with every Chern–Simons form written out sector by sector is in Appendix B.

Theorem 5.5 (T-BI6-6.3: BI-6 local anomaly cancellation). *If the total one-loop anomaly polynomial factorizes as $I_8^{\text{tot}} = \frac{1}{2}(X_4^{\text{BI6}})^2$, and if the two-form B_2 transforms as $\delta B_2 = -\omega_2^{(1)} + d\Lambda_1$, then the total effective action is gauge invariant:*

$$\boxed{\delta(\Gamma_{1\text{-loop}} + S_{\text{GS}}) = 0.} \quad (46)$$

Proof. The proof proceeds in four steps.

Step 1: Chern–Simons potential. Since X_4 is a closed four-form built from characteristic classes, there exists a Chern–Simons three-form $\omega_3^{(0)}$ satisfying

$$d\omega_3^{(0)} = X_4. \quad (47)$$

For BI-6, $\omega_3^{(0)}$ is the linear combination of the gravitational, $\text{SU}(5)$, $U(1)_B$, and $U(1)_{Q_i}$ Chern–Simons forms with the same coefficients $(1, -1, -2/5, -2/5)$.

Step 2: Seven-form and six-form descendants. Define the seven-form $I_7^{(0)} = \frac{1}{2}\omega_3^{(0)} \wedge X_4$. Then $dI_7^{(0)} = \frac{1}{2}X_4 \wedge X_4 = I_8$, as required. Under an infinitesimal gauge/Lorentz variation, the Chern–Simons form varies as $\delta\omega_3^{(0)} = d\omega_2^{(1)}$, where $\omega_2^{(1)}$ is the descent two-form. Because $\delta X_4 = 0$ (characteristic classes are gauge-invariant), the consistent anomaly six-form is

$$I_6^{(1)} = \frac{1}{2}\omega_2^{(1)} \wedge X_4. \quad (48)$$

Step 3: One-loop anomalous variation. The one-loop effective action has the anomalous variation

$$\delta\Gamma_{1\text{-loop}} = \frac{\kappa}{2} \int_{M_6} \omega_2^{(1)} \wedge X_4. \quad (49)$$

Step 4: Green–Schwarz counterterm. The counterterm is $S_{\text{GS}} = \frac{\kappa}{2} \int_{M_6} B_2 \wedge X_4$. Using $\delta B_2 = -\omega_2^{(1)} + d\Lambda_1$ and $\delta X_4 = 0$:

$$\delta S_{\text{GS}} = \frac{\kappa}{2} \int_{M_6} (-\omega_2^{(1)}) \wedge X_4 = -\frac{\kappa}{2} \int_{M_6} \omega_2^{(1)} \wedge X_4. \quad (50)$$

(The $d\Lambda_1$ term integrates to $\frac{\kappa}{2} \int_{\partial M_6} \Lambda_1 \wedge X_4$, which vanishes on a closed manifold. The descent is a bulk statement: on the intended class the Euclidean problem is posed with the collars of clause

(C1) of Definition 6.1 or on the doubled interval, and what the ends of the interval carry — the boundary-localized anomaly of the split spectrum and the tensor’s own boundary variation — is not computed by it. The four-dimensional $U(1)_B$ gauge parameter is removed at the ends by the Dirichlet condition of clause (C9), Section 5.7, and the level-by-level inflow check on the terminated interval is Open Problem O13.)

Adding (49) and (50):

$$\delta\Gamma_{1\text{-loop}} + \delta S_{\text{GS}} = +\frac{\kappa}{2} \int \omega_2^{(1)} \wedge X_4 - \frac{\kappa}{2} \int \omega_2^{(1)} \wedge X_4 = 0. \quad (51)$$

This completes the proof. ■

Corollary 5.6 (BRST exactness and Slavnov–Taylor identity). *Under the hypotheses of Theorem 5.5, the anomaly cocycle is BRST-exact in the enlarged field space containing B_2 , and the gauge-fixed effective action satisfies the homogeneous Slavnov–Taylor identity.*

Remark 5.7 (Scope of the structural theorem). Theorem 5.5 is a statement about the *perturbative local* anomaly—the eight-form I_8 and its cancellation by a two-form counterterm through descent. A complete anomaly-freedom claim in the modern sense requires additional global checks: the exponentiated path integral must be single-valued under large gauge transformations, the self-dual tensor charge lattice must be properly quantized, and the Atiyah–Patodi–Singer eta invariant on bounding seven-manifolds must be consistent. These global questions have been addressed systematically for $\mathcal{N} = (1, 0)$ supergravity by Monnier and Moore (Monnier and Moore, 2019) and, more recently, by Dierigl and Tartaglia (Dierigl and Tartaglia, 2025).

For GG-6, the standard partial checks are recorded in Section 5: $\pi_6 = 0$ for all gauge factors, $\Omega_7^{\text{Spin}}(\text{pt}) = 0$, and the chiral-tensor signature phase is settled by the Wu–Chern–Simons quadratic refinement (manifestly trivial on the $\Omega = 50$ branch, where $n_{\text{SD}} = 200 \equiv 0 \pmod{8}$; Appendix H). The complete global theorem, including the Wu–Chern–Simons / differential-cohomology refinement on the physical intended GG-6 background class, is established in Section 6 and Appendix I. The structural theorem and the model-level benchmark are not affected by this distinction: the local polynomial factorization and coefficient matching are exact statements that hold independently of the global completion.

5.7 The boundary mixed anomalies in four-dimensional language

A reader trained in four-dimensional effective field theory will ask the anomaly question in a different dialect than the descent proof above, and the question deserves an answer in its own words. Gauging baryon number in the Standard Model *alone* is famously obstructed: with the observed fermion content, in the left-handed Weyl convention, the mixed triangle coefficients per generation are

$$\mathcal{A}[U(1)_B\text{--}SU(2)_L^2] = \sum_{\text{doublets}} BT(R) = 3_c \cdot \frac{1}{3} \cdot \frac{1}{2} = +\frac{1}{2}, \quad \mathcal{A}[U(1)_B\text{--}U(1)_Y^2] = \sum BY^2 = -\frac{1}{2}, \quad (52)$$

while the remaining candidates vanish identically for Standard-Model content: $\mathcal{A}[\text{U}(1)_B\text{-grav}^2] = \sum B = 0$ (baryon number is vector-like on the quarks), $\mathcal{A}[\text{U}(1)_B^3] = 0$, and $\mathcal{A}[\text{U}(1)_B^2\text{-U}(1)_Y] = 0$. Exactly two mixed triangles survive, equal and opposite, summing over three generations to $\pm\frac{3}{2}$. The conventional price of gauging $\text{U}(1)_B$ is therefore a set of new chiral fermions (“anomalons”) charged to cancel (52) — new matter invented for bookkeeping, with masses and signatures of its own.

BI-6 pays no such price, and the reason is already proved above; this subsection only translates it. The boundary chiral spectrum of GG-6 is not the Standard Model with baryon charges appended by hand: it descends from unsplit $\text{SU}(5)$ multiplets and the OGN-locked charge lattice (Section 7), and the microscopic anomaly polynomial of that spectrum *factorizes to rank one* (Theorem 5.1 and the microscopic remark following it) — the cross-blocks of the matrix M_{BI6} in (43), the $(F_B^2, \text{Tr } F_5^2)$ and $(F_B^2, \text{tr } R^2)$ entries, are precisely the bulk ancestors of the mixed anomalies the four-dimensional reader is worried about, and the single counterterm $\int B_2 \wedge X_4$ cancels those ancestors in the bulk (Theorem 5.5; the inflow of Callan and Harvey, 1985). On the admissible class no cell flux carries them to the boundary as a gauge anomaly — the Dirichlet condition stated below removes the four-dimensional $\text{U}(1)_B$ gauge parameter at the ends — so the triangles are not canceled on the boundary by the counterterm; they remain, as the next paragraph states, the anomalous divergence of a global current, and nothing has to cancel them. The statement that the bulk cross-blocks split across the Standard-Model subgroups $\text{SU}(2)_L \times \text{U}(1)_Y \subset \text{SU}(5)$ in exactly the ratios required by (52) is the content of the rank-one factorization applied to the split boundary spectrum; the explicit split-multiplet bookkeeping lives with the model-level benchmark of Section 7, and the global caveats of Remark 5.7 apply unchanged.

In four-dimensional language the mechanism has a familiar face, and the mode analysis of Section 5.9 identifies which face it is. One candidate was the four-dimensional Green–Schwarz mechanism, as realized for anomalous $\text{U}(1)$ s in string constructions (Dine, Seiberg, and Witten, 1987, Ibáñez, Rabadán, and Uranga, 1999): the light boundary remnant of B_2 would shift under a $\text{U}(1)_B$ gauge transformation and be eaten. That shift, however, requires a background flux through the (r, θ) cell — $\Phi_B = (\vartheta_B(K) - \vartheta_B(0))/2\pi$, the difference of the $\text{U}(1)_B$ Wilson line between the two ends of the throat — and on the admissible class of Definition 6.1, with the variational boundary conditions and no boundary datum for the Abelian Wilson lines, $\Phi_B = 0$ identically: the remnant does not shift and is not eaten. The position this leaves is unusual enough to state directly: BI-6 keeps the six-dimensional Green–Schwarz structure in full — the modified Bianchi identity $dH_3 = X_4$ and the counterterm $\int B_2 \wedge X_4$, a member of the $B \wedge F$ family of topological couplings, in which a mass is generated with no condensate and no physical scalar — and declines only its four-dimensional descendant, the step in which the surviving remnant is eaten. The descendant is declined on a computation and not on a preference: Φ_B vanishes identically on the admissible class, so that step is unavailable here rather than unwanted. What removes the obstruction instead is a boundary condition. The four-dimensional zero mode of $A_{B,\mu}$ is removed by a Dirichlet condition, so no massless baryonic vector exists, no local gauge parameter survives at the boundary hosting the anomalous zero modes, and the mixed triangles (52) remain what they are in the Standard Model: the anomalous divergence of a *global* current, violated by electroweak instantons and by nothing

else (the level-by-level consistency of the *massive* baryonic tower’s coupling to this current is the registered check of Open Problem O13, Section 17). Three consequences follow. *First*, no anomalous fermions exist anywhere in the spectrum — the ledger cancels its own books, and the boundary condition does the terminating. *Second*, $U(1)_B$ survives on the boundary as an exact perturbative selection rule, so the proton-stability falsifier F-BI6-1 stands, conditional on the holonomy-descent step fenced in (Arisaka, 2026, A2-GG6). *Third*, the massive vector itself is not an abstraction: it is the leptophobic Z'_B of the lock cluster — the first Kaluza–Klein state of the Dirichlet-terminated $A_{B,\mu}$, at 3.85 TeV, degenerate with the graviton and, on the both-ends condition of clause (C9), unproduced by the endpoint quarks at tree level — whose collider phenomenology is the business of GG-P1-KK (Arisaka, 2026, P1-KK), the place where this subsection’s bookkeeping meets a detector.

5.8 The terminated-seam capacity theorem: the coupling of the boundary-terminated $U(1)_B$

The preceding subsection established that the $U(1)_B$ vector’s zero mode is removed by a boundary condition, its lightest state sitting at the lock scale. One number remained open across the cohort: the coupling at which the massive vector talks to the boundary baryon current. GG-P1-KK (Arisaka, 2026, P1-KK) carried that number to its honest boundary — the generator norm is theorem-grade, and the residual freedom is a single discrete rule, registered there as L-KK-1 (“state and prove the capacity readout for a Stueckelberg-terminated Abelian seam”), with four candidate readouts bracketed — the registration’s wording predates the identification of the termination as a boundary condition (Section 5.9); the readout it demands is unchanged. The rule is proved here, because every structure it needs is this paper’s own: the rank-one lattice, the X_4 integrality datum, and the inflow that feeds the Arisaka four-currents lock. The execution record — the registered computation PS-5thF-1 of the P-series fifth-force paper, where the consequences were printed before the attempt — ships beside that paper’s source with its verification script.

Theorem 5.8 (T-BI6-6.4: Terminated-seam capacity readout for $U(1)_B$). *On the physical intended GG-6 background class (Definition 6.1), the boundary coupling of the boundary-terminated baryonic seam is its linear ledger share of the cell stiffness:*

$$\alpha_B^{-1} = \frac{\text{Tr}(B_0^2)}{\text{Tr}(Y_0^2)} \alpha_Y^{-1} = \frac{L - N}{L} \alpha_Y^{-1} = \frac{2}{5} \alpha_Y^{-1}. \quad (53)$$

Equation (53) is a tree-level matching condition at the seam, in the unmixed basis, conditional on the imported stiffness theorems it cites and on the identification of the boundary trace with the bulk seam’s share (the stiffness clause named in Step 1 of the proof); the one-loop B – Y kinetic mixing conceded at (33) and the ordinary running below the seam are not addressed by it (Objection O-BI6-54, Open Problem O13). At the idealized Z -attractor values of (Arisaka, 2026, A9-SM) ($\alpha^{-1}(M_Z) = 2^J = 128$, $\sin^2 \theta_W = 25/108$) it is exact and rational,

$$\alpha_B^{-1} = \frac{2}{5} \cdot 2^J \cos^2 \theta_W = \frac{5312}{135} = 39.348, \quad g_B = \sqrt{4\pi\alpha_B} = 0.565 : \quad (54)$$

the third entry of GG-P1-KK’s menu, now derived. (Under the cohort’s own running correction

$\alpha^{-1}(M_Z) = 2^J - N^2/2^J = 127.93$, the share shifts to $\alpha_B^{-1} = 39.33$; $g_B = 0.565$ is stable at the three figures printed.)

Proof. Three steps, each consuming a published structure.

Step 1 (linear stiffness law). By the capacity identification of (Arisaka, 2026, A9-SM) (T-SM-27.1), the gauge-invariant ledger variable of an Abelian seam is the stiffness $C = \alpha^{-1}$, and the local ledger is a quadratic form on the reduced conductance variables of the primitive cell — one variable per state of the defining **5**. Before the $3 \oplus 2$ split the cell carries the unbroken $SU(5)$ of the unified block, and an $SU(5)$ -invariant quadratic form on the **5** is proportional to the identity, exactly. The stiffness of the seam generated by a diagonal charge Q is therefore $S_Q = s_0 \text{Tr}(Q^2)$, with one constant s_0 for every seam of the cell: the norm enters *linearly*. That argument fixes the normalization of generators inside $\mathfrak{su}(5)$; B_0 is the boundary charge on the split **5**, with a trace part outside that algebra, and reading its trace $\text{Tr}(B_0^2)$ as the bulk seam's share by the same constant s_0 is the imported stiffness clause of this proof, named here. Because a bulk factor that commutes with $SU(5)$ is trace-orthogonal to hypercharge on every block, no tree-level B – Y mixing exists above the split; the mixing arises at one loop below it, from the boundary fermions with $\sum BY = 2/3$ per generation. (The quadratic form of the norm belongs to *weight* observables — the partition fractions, where (Arisaka, 2026, A9-SM) squares it — never to stiffness variables; A9 draws this line inside its own two proofs.)

Step 2 (no private ledger). The termination cannot alter the share. First, the tensor sector is rank one (Theorem 5.1; the tensor clause of Definition 6.1): a seam with a stiffness history of its own would constitute a second, independently normalized ledger channel, which does not exist. Second, the $U(1)_B$ share is the coefficient $\gamma = 2/5$ of F_B^2 in X_4 , fixed by the primitive integrality condition $(s, t) = (2, 5)$ — a topological datum of the differential-cocycle lift, invariant under lawful refinement (POS-5): the dyadic cascade that carries the cell's stiffness from 1 to 2^J multiplies the account and cannot reapportion the shares. Third, at the boundary the lock closes the last route: by the Arisaka equation (Arisaka, 2026, A7-Qi4, T-QUA; Arisaka, 2026, A9-SM, T-SM-10.3) the baryonic current is not an independent channel but a projection of the same rank-one inflow as the electric current, with a single conversion factor. A B -stiffness frozen at a private cascade stratum $n < J$ would put a relative dyadic factor 2^{J-n} between the two projections — a second conversion factor, which T-QUA forbids. The termination therefore acts where the preceding subsection put it — in the mass channel alone, removing the zero mode and leaving the first Kaluza–Klein note at M_{lock} — and nowhere else; the coupling share is untouched.

Step 3 (evaluate). With both members of the split-generator pair sharing one s_0 , $\alpha_B^{-1}/\alpha_Y^{-1} = S_B/S_Y = \text{Tr}(B_0^2)/\text{Tr}(Y_0^2) = (L-N)/L = 2/5$, which is Eq. (53); evaluating at the Z -attractor values gives Eq. (54). ■

Remark 5.9 (The menu, and where the quadratic lives). The four readouts GG-P1-KK bracketed were the norm entering linearly or quadratically, on either side of the capacity identification. Step 1 selects linear; Step 1's identification selects the side (the seam owning less of the cell's stiffness is the softer one, and couples more strongly — the direction ordinary unified normalization also takes, $\alpha_i^{-1} \propto \text{Tr}(T_i^2)$). The quadratic candidates were not noise, but no laboratory observable owns the square: portal *amplitudes* are linear in the X_4 coefficients ($1 : \frac{2}{5}$, the baryonic entry the smaller),

and the baryonic entry is carried by a vector whose lightest state is the 3.85 TeV note, below which there is no baryonic channel for any receiver to read; the two-channel power ratio formerly advertised on this arithmetic is retired. An outside-menu readout would require a second inflow channel, which Theorem 5.1 excludes. No scale remains open in this sector: with the termination a boundary condition there is no Stueckelberg scale and no eaten decay constant *at the zero mode* — the vector’s mass is the pipe’s own note, M_{lock} , and it mediates no macroscopic force. The qualifier is not a hedge. It is the zero mode that the Dirichlet condition removes; the massive levels above it behave as the levels of any compactified gauge field do, each taking its longitudinal state from the corresponding mode of $A_{B,r}$, so the massive baryonic tower is Stueckelberg-massive level by level. That structure is what Open Problem O13 checks: the inflow bookkeeping of the massive tower is the consistency of the level-by-level pairing with the anomaly inflow, which is why it is an expectation and not yet a theorem. The scale question that survives belongs to the tensor remnant itself, its canonical decay constant (Open Problem O14 of Section 17), not to the vector.

The unit on which the note is read should be stated once, because the cohort’s papers have been read as using two. With the Dirichlet tower’s first zero at $x_1 = 3.8317$ and the note at M_{lock} , the curvature unit is $ke^{-K} = M_{\text{lock}}/3.8317 = 1.005$ TeV — the unit on which the collider companion places its spin-2 anchor (Arisaka, 2026, P1-KK), so that the baryonic vector and the graviton sit at the lock together on one unit. The Neumann towers — color and electroweak — have their free-field first level at 2.4499 curvature units on that same unit, 2.46 TeV, a factor 1.56 below the lock; the companion’s placement of the color octet *at* the lock is its composite-hardened reading of the same geometry (its Scenario B), a statement about the strong dynamics of the boundary sector and not a second curvature unit. GG-A9-SM’s tree-level oblique parameter is printed in both readings (Arisaka, 2026, A9-SM): $T = 1.7$ with the electroweak first level at the lock and free-field overlaps — the unit on which that level *is* the lock would be 1.57 TeV — and $T = 4.2$ at the free-field level on this unit, both at kill grade in its Open Problem O22. Neither reading moves the baryonic vector: a Dirichlet mode sits at the lock on this unit whatever the Neumann towers do, which is the point of Open Problem O15 below.

5.9 Self-duality of the Green–Schwarz tensor

Proposition 5.10 (Necessity of anti-self-duality). *A six-dimensional theory whose chiral anomaly is canceled by a single Green–Schwarz tensor requires the tensor to be anti-self-dual at the quantum level. No spectrum of ordinary Weyl fermions alone can produce the required gravitational coefficients.*

Proof. Every Weyl fermion in six dimensions contributes to the anomaly polynomial a gravitational piece proportional to $(7p_1^2 - 4p_2)$, regardless of its gauge representation. The target polynomial $\frac{1}{2}X_4^2 = \frac{1}{2}(\text{tr } R^2 - \dots)^2$ contains only $(\text{tr } R^2)^2 \propto p_1^2$ and no independent p_2 term. Since fermions always produce p_2 with a fixed ratio $-4/7$ relative to p_1^2 , fermions alone can never cancel the p_2 term while retaining a nonzero p_1^2 term. An anti-self-dual tensor contributes gravitational anomaly through the Hirzebruch L -polynomial with coefficients $(p_1^2 - 7p_2)/360$ per tensor (see Appendix H for the explicit formula), which has a *different* p_2/p_1^2 ratio than fermions. This breaks the degeneracy and allows both the p_1^2 and p_2 equations to be satisfied simultaneously. ■

The tensor’s four-dimensional remnant. On the canonical warped cell $\mathcal{M}_4 \times I_r \times S_\theta^1$ the chiral tensor leaves exactly one light pseudoscalar: the flat $B_{\mu\nu}$ mode and its duality partner $B_{r\theta} \propto e^{2r}$ are one degree of freedom, with kinetic norm $\pi(e^{2K} - 1)$; the flat $B_{r\theta}$ mode of norm $\pi(1 - e^{-2K})$ is the opposite chirality’s zero mode and is absent. Under the reflection $r \rightarrow -r$ the assignment $B_{\mu\nu}, B_{\mu\theta}$ even and $B_{\mu r}, B_{r\theta}$ odd is the one compatible with chirality (the orientation form flips with the reflection); a decoupled vector zero mode $B_{\mu\theta}$ of norm $2\pi K$ also survives these parities, couples to X_4 only through Kaluza–Klein modes, and is recorded as an open exposure. The r -constant Wilson-line fluctuation of the gauge field $A_{B,\theta}$, which survives the Neumann condition of clause (C9) of Definition 6.1, is registered on the same terms: on the intended class its holonomy is fixed background data, and its fluctuation is a second open exposure carried here beside the tensor’s vector mode. That exposure now carries weight elsewhere in the cohort, and the weight should be named. In a flat-circle model the one-loop potential of the endpoint quarks does not select the ledger’s holonomy, $M = 1$ in the lock-scale units: its minimum sits at $M = 7, 8$, tied, for periodic quarks and at $M = 0$ for antiperiodic ones. The ledger’s value therefore stands on this clause — background data on the admissible class — and not on a dynamical selection this paper or any companion has performed. Two consequences rest on it: GG-A10-Cosmos’s two Hubble rows, held at the derived-starred grade (Arizaka, 2026, A10-Cosmos), and the selection half of GG-A5-GDOS’s Open Problem O6, the derivation of ζ_Λ (Arizaka, 2026, A5-GDOS). A computation that selected a different holonomy would not falsify BI-6; it would move the ledger’s value off the clause, and those consumers with it. A third follows from the Dirichlet condition itself. With the gauge parameter fixed at both ends of the interval, the r -integral $\int_0^K A_{B,r} dr$ cannot be gauged away: its fluctuation is a real four-dimensional scalar — the mode of $A_{B,r}$ orthogonal to the gauge directions, whose profile e^{2r} on the canonical metric weights the infrared end — massless at tree level and lifted only by the loop potential of the fields charged under $U(1)_B$. The same holds for $U(1)_{Q_i}$. On the intended class both interval Wilson lines are fixed background data; the masses and couplings of their fluctuations are open, and the collider companion registers them among its own open items (Arizaka, 2026, P1-KK). An infrared-end-only termination — Dirichlet at $r = K$ alone — would leave no such mode, at the price of moving the first baryonic Kaluza–Klein level off the lock scale by about one percent; this paper keeps the both-ends condition of clause (C9) and prints the scalar rather than the silence. One convention this subsection must eventually fix, and does not yet: the orientation of the anti-self-duality condition relative to the warp direction (a convention within the anti-self-dual sector, not the choice of chirality itself, which the anomaly fixes), which selects between the two norms above and thereby moves the remnant’s canonical decay constant by e^K — the content of Open Problem O14 (Section 17).

6 Complete global anomaly freedom of BI-6 under GG-6

Section 5 established that the BI-6 factorization cancels the *infinitesimal* anomaly by descent. That is necessary but not sufficient. A modern anomaly-freedom proof must also address *large* gauge transformations and diffeomorphisms, which can introduce a global phase invisible to the infinitesimal analysis. The physical question is: if one transports the quantum theory around a

non-contractible loop in the space of gauge-equivalent backgrounds, does the partition function return to itself, or does it pick up an anomalous phase? Answering this requires the machinery of mapping tori, eight-dimensional fillings, and the Wu–Chern–Simons refinement of the self-dual tensor sector. The present section states the precise background class, formulates the global theorem, and gives the proof strategy; the detailed lemmas are carried out in Appendix I.

6.1 The physical intended GG-6 background class and the main theorem

The global theorem is only as strong as the class of backgrounds on which it is stated. For that reason the intended GG-6 class is written explicitly in the main text rather than left implicit or deferred. These nine clauses define the geometric and topological class of six-dimensional backgrounds used by GG-6.

Definition 6.1 (Physical intended GG-6 background class). The physical intended GG-6 background class Admissible consists of data

$$(M_6, \mathcal{P}_5, L_B, L_{Q_i}, \check{B}_2)$$

satisfying the following clauses.

(C1) **Stage clause.** The Lorentzian bulk has the GG-6 product stage

$$M_6 = M_4 \times I_r \times S_\theta^1,$$

with M_4 spin and the two structural directions r and θ distinguished. For Euclidean anomaly problems one uses APS-compatible collars at the ends of I_r or doubles the interval to obtain the closed structural torus

$$T_{r\theta}^2 = S_r^1 \times S_\theta^1.$$

(C2) **Gauge clause.** The exact gauge group is

$$\mathrm{SU}(5) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_{Q_i}.$$

No extra gauge factors are admitted in the intended class.

(C3) **Tensor clause.** There is a single primitive X_4 -coupled tensor channel, equivalently a rank-one unimodular tensor lattice $\Lambda \cong \mathbb{Z}$ for the Green–Schwarz sector, with characteristic vector $a = 1$. The same $a = 1$ is the characteristic element that defines the unique Wu–Chern–Simons quadratic refinement of the tensor sector (Theorem 6.13). (The gauge-singlet completion tensors of Section 7 are X_4 -decoupled spectators; they enter the global analysis only through their signature-type phase, Appendix H.)

(C4) **BI-6 clause.** The four-form entering the modified Bianchi identity is

$$X_4 = \mathrm{tr} R^2 - \mathrm{Tr} F_5^2 - \gamma(F_B^2 + F_{Q_i}^2), \quad \gamma = \frac{s}{t} \in \mathbb{Q}_{>0}, \quad \mathrm{gcd}(s, t) = 1,$$

subject to the primitive integrality condition

$$\hat{X} := \frac{t}{8\pi^2} X_4 \in H^4(M_6, \mathbb{Z}).$$

BI-6 is the point $\gamma = 2/5$, that is $(s, t) = (2, 5)$. The clause is stated at general γ because every consistency theorem of the present Part is proved at that generality; Remark 6.3 explains why this is the honest statement rather than a weaker one. Thus the BI-6 class is primitive and admits a differential-cocycle lift.

- (C5) **Structural-holonomy clause.** The structural torus $T_{r\theta}^2$ is fixed as part of the background data. Allowed transformations preserve the product/collar structure and do not introduce modular monodromy of the (r, θ) torus.
- (C6) **Allowed large transformations.** Large diffeomorphisms are identity on the structural collars (or isotopic to identity after doubling) and product-preserving on $T_{r\theta}^2$. Gauge clutching on the structural torus is therefore null-homotopic in the intended class; the same clause excludes mixed clutching (gauge classes in $H^2(M_4; \mathbb{Z}) \otimes H^1(T_{r\theta}^2; \mathbb{Z})$) and Abelian winding transformations on the structural circles. Transformations outside this class were formerly the unrestricted case of Open Problem O10; Theorem 6.9 (Subsection 6.3) now covers them, and the clause survives as a description of the physical intended backgrounds rather than as a hypothesis of the anomaly theorem.
- (C7) **Physical base clause.** The Euclideanized compact four-manifolds used in the intended GG-6 class satisfy

$$H^1(M_4; \mathbb{Z}) = 0, \quad H^3(M_4; \mathbb{Z}) = 0.$$

Equivalently, the connected physical bases carry no independent Abelian winding sector and no independent $SU(5)$ gauge-transformation class. For noncompact Minkowski backgrounds, large transformations are taken with compact support so that the same null-homotopy conclusion holds after compactification.

- (C8) **Pullback-bundle clause.** The principal $SU(5)$, $U(1)_B$, and $U(1)_{Q_i}$ bundles are pullbacks from the base and are topologically trivial on the structural torus. The BI-6 differential cocycle is likewise a pullback/refinement of this base data.
- (C9) **Abelian boundary-condition clause.** On the interval I_r the four-dimensional components $A_{B,\mu}$ carry a Dirichlet condition removing their zero mode, and $A_{B,\theta}$ a Neumann condition — the pair chosen as the variational data of the interval, not derived from a parity — so the $U(1)_B$ Wilson line carries no gradient across the throat ($\Phi_B = 0$). The r -constant Wilson-line mode of $A_{B,\theta}$ survives the Neumann condition as the circle's $U(1)_B$ holonomy; on the intended class it is fixed background data — a background gauge datum fixed with the pullback-bundle data of clause (C8) — and its fluctuation is an exposure recorded beside the tensor's vector mode (Section 5.9). The $U(1)_{Q_i}$ components carry the same pair of conditions, and so do the twelve coset directions of $SU(5)$: a coset vector with an endpoint profile would couple an endpoint quark to an endpoint lepton,

a boundary operator of nonzero baryon number that the descended boundary charge forbids, so the X and Y directions are endpoint-decoupled, their first level at M_{lock} beside the baryonic vector and pair-produced above $2M_{\text{lock}} = 7.7$ TeV; $SU(5)$ is broken on the boundary by the split, not by a bulk Higgs. Because the Dirichlet condition fixes the gauge parameter at both ends, the integral of $A_{B,r}$ across the interval is gauge-invariant as well: this interval Wilson line is likewise fixed background data on the intended class, and its fluctuation — a four-dimensional scalar, massless at tree level — is registered with the circle holonomy's in Section 5.9. No boundary datum pins an Abelian Wilson line to distinct values at the two ends. (This clause is what makes the termination of Section 5.7 a boundary condition rather than a Green–Schwarz shift; see the mode analysis of Section 5.9.)

These clauses are not ad hoc additions. They simply spell out the class of backgrounds already used by the GG-6 stage itself: fixed product structure, exact gauge content, one primitive tensor, no hidden torus monodromy, and physically relevant four-dimensional bases without extra topological gauge sectors.

Theorem 6.2 (T-BI6-7.2: Complete anomaly freedom of BI-6 on the physical intended GG-6 class). *On Admissible, the combined phase of the chiral determinant and the exponentiated Wu–Chern–Simons / differential-cohomology lift of the BI-6 counterterm is globally well-defined and gauge-invariant. Equivalently, BI-6 is completely anomaly-free, both locally and globally, on the physical intended GG-6 background class.*

Proof strategy and physics intuition. The proof has four logically separate pieces.

Step 1: local descent. The infinitesimal anomaly is the six-dimensional descent from $I_8 = \frac{1}{2}X_4^2$. This is the part proved already in Section 5: the one-loop consistent anomaly is canceled exactly by the local Green–Schwarz counterterm.

Step 2: seven-dimensional holonomy. A global anomaly is not a failure of the local descent equations; it is a phase picked up when one transports the quantum theory around a non-contractible loop in the space of backgrounds. That phase is encoded by the holonomy of the anomaly line on the seven-dimensional mapping torus, and is evaluated on an eight-dimensional filling by the Atiyah–Patodi–Singer / Dai–Freed formalism (Dai and Freed, 1994). The global problem is therefore a $6D \rightarrow 7D \rightarrow 8D$ consistency ladder.

Step 3: globally defined tensor counterterm. The local expression $\frac{1}{2} \int B_2 \wedge X_4$ is insufficient on nontrivial topology. The correct global object is the Wu–Chern–Simons / differential-cohomology refinement associated with the integral BI-6 class $\hat{X}$ and the characteristic vector of the rank-one tensor lattice (Hopkins and Singer, 2005, Monnier and Moore, 2019). The quadratic refinement is what makes the self-dual tensor sector single-valued rather than multi-valued.

Step 4: automatic admissibility on Admissible. The remaining task is to show that the hypotheses used in the global theorem are automatic on the intended GG-6 class. The detailed

lemmas are proved later in the appendices: π_6 vanishes for $SU(5) \times U(1)_B \times U(1)_{Qi}$, the BI-6 class admits an integral lift, the Wu-shifted quadratic functional is properly quantized, and the product/collar mapping tori bound compatible eight-manifolds. Once these facts are assembled, the global phase cancels exactly. ■

Remark 6.3 (The theorem holds for a family, and why that is the right statement). Theorem 6.2 has been stated and proved on Admissible with the Abelian coefficient carried as a symbol. It is worth saying plainly what that buys and what it costs, because a reader may take the generality as a concession.

What the proof actually consumes. Walk the chain and ask, at each step, which property of γ is used. The local descent (Theorem 5.5) needs only that X_4 is closed and gauge-invariant; the coefficient never appears in the argument, only in the name of the object. The rank-one criterion (Theorem 5.1) is linear algebra over the four quadratic invariants and is blind to their coefficients. The integral lift (Lemma I.16) uses exactly one property of γ : that it is rational, so that some integer clears its denominator. The Wu–Chern–Simons quantization needs the spin condition and the integrality of x , and nothing else — the lemma says so in as many words. The vanishing of π_6 is a fact about $SU(5) \times U(1)^2$. The bounding theorem is a fact about the topology of the base and the structural torus.

Not one of these six steps can see the value of γ . That is not a weakness of the proof; it is a description of what kind of theorem it is.

Why stating the family is stronger than stating the point. Three reasons, in increasing order of importance. First, it is simply a stronger theorem: a statement about a one-parameter family implies the statement at any point of it. Second, it is robust — the value $2/5$ is fixed elsewhere in this paper, by the trace-ratio identity of Theorem 8.1, and if that identity were ever revised the anomaly theorem would not move an inch. The consistency of the bulk does not wait on the arithmetic of the ledger. Third, and most usefully, it locates the burden of proof correctly.

The objection this invites, and the answer. A reader may say: if anomaly freedom holds for a whole family, then anomaly freedom does not single out BI-6, and the paper’s claim to rigidity is empty. The first half of that is correct and we state it ourselves. The second half does not follow, and the reason is worth being exact about. Anomaly freedom is a *consistency test*. Consistency tests are passed by families — that is what makes them tests rather than derivations, and it is equally true of the 1984 ten-dimensional mechanism, which admits both $SO(32)$ and $E_8 \times E_8$ and does not thereby become vacuous. What is claimed to single out BI-6 was never the anomaly theorem. It is the constitution: AX-4 (MSMF) fixing the single-tensor architecture and the equal-Abelian treatment, and the trace-ratio identity fixing the Abelian fraction. Those claims stand or fall on their own merits, in the open, and this Part does not lend them support it does not have.

Figure 5 summarizes the logic of the global theorem as a dimensional ladder, together with the two facts that close it on the intended class; Subsection 6.3 then removes the intended-class restriction itself.

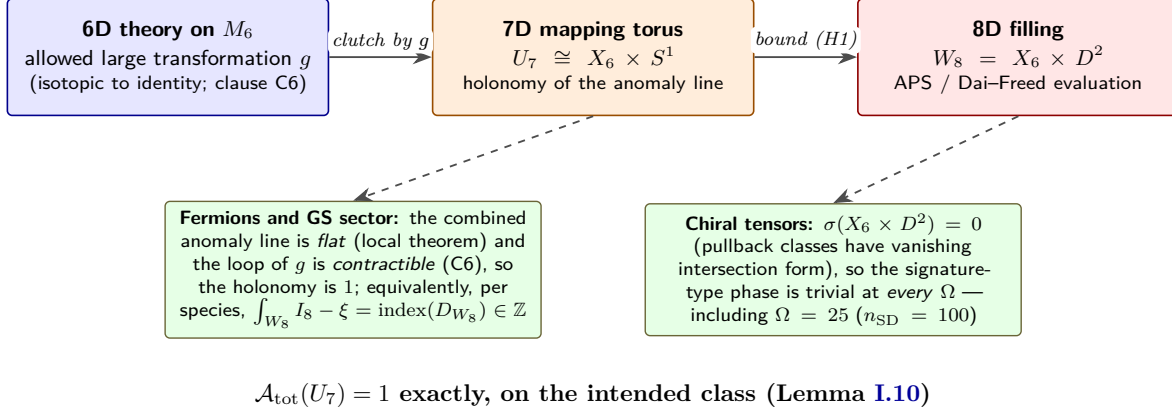


Figure 5: The global-anomaly dimensional ladder 6D → 7D → 8D and the two facts that close it on the intended GG-6 class. An allowed large transformation g (isotopic to the identity by clause (C6)) clutches the 7D mapping torus $U_7 \cong X_6 \times S^1$, which bounds the product filling $W_8 = X_6 \times D^2$ with all data extending (H1). For the fermion and Green–Schwarz sectors the combined anomaly line is flat by the local theorem, and the loop is contractible, so its holonomy is exactly 1 — equivalently, each species’ Dai–Freed factor is $\exp(2\pi i \cdot \text{index}) = 1$ by the Atiyah–Patodi–Singer index theorem (Atiyah *et al.*, 1975). For the chiral-tensor sector, $\sigma(X_6 \times D^2) = 0$, so the signature-type phase is trivial at every Ω . Together these give Lemma I.10: the global phase on the intended class is computed, not merely constrained.

The theorem is complete on the physical intended GG-6 class used by the series. What is not claimed is a broader theorem for every arbitrarily enlarged non-supersymmetric six-dimensional category with extra monodromies, extra gauge sectors, or unrestricted topology. That broader universalization would require a separate argument.

Remark 6.4 (Scope of the admissible class). We state explicitly what the admissible class Admissible does and does not test. Clauses (C5)–(C8) exclude non-trivial torus monodromies, non-simply-connected bases, non-pullback bundle topologies, and large gauge clutching on the structural directions. On the remaining class, the mapping tori are relatively simple—they factorize as products, the gauge clutching is null-homotopic by construction, and the only nontrivial check is the Wu-shift integrality, which follows from a standard formula on spin eight-manifolds.

It is worth naming the hardest cases in the modern global-anomaly literature, because they are the ones a reader will ask about: backgrounds with non-trivial torsion in H^3 , exotic gauge-bundle topology over non-simply-connected manifolds, or complicated clutching producing topologically non-trivial mapping tori. Theorem 6.9 has since removed that concession for the GG-6 gauge content: those cases are covered, because the relevant spin bordism groups vanish (Lemma 6.6) and genuinely non-bordant mapping tori do not exist for this gauge group. A specialist in six-dimensional global anomalies will still correctly observe that the vanishing is a special property of the GG-6 gauge content — an arbitrary gauge packet would not enjoy it — and that observation is now the precise sense in which the intended class is favorable.

The response from the axiomatic perspective is that this restriction is a *consequence of the physical design*, not an ad hoc retreat. The same clauses that simplify the global anomaly problem—the product structure, the fixed gauge content, the single primitive tensor, the absence of extra

topological sectors—are the clauses that make GG-6 finite, predictive, and falsifiable. A theory that required arbitrary base topology and arbitrary bundle classes would also require arbitrary parameters and an uncontrolled landscape (Bousso and Polchinski, 2000), which is precisely what the constitution (AX-4/POS-6) forbids. In this sense, the restrictiveness of Admissible is a feature that delivers phenomenological predictivity — but, after Theorem 6.9, it is no longer what delivers global consistency.

The broader question—whether a theory with the BI-6 gauge content and coefficient vector but *unrestricted* topology remains globally consistent—was explicitly left open in the conditional form of the theorem, and is now answered affirmatively for the GG-6 gauge content in Subsection 6.3: the relevant spin bordism groups of BG vanish, every closed spin seven-datum bounds, and the phase evaluates to one. What the GG-6 chain does not claim, and what remains a well-posed invitation to the community, is the same analysis for arbitrary gauge packets and multiplet structures in the general non-SUSY setting of Monnier–Moore (Monnier and Moore, 2019) and Dierigl–Tartaglia (Dierigl and Tartaglia, 2025).

6.2 The sphere-background case, as a corollary

Proposition 6.5 (Absence of Witten-type global anomalies). *For the GG-6 bulk gauge group $SU(5) \times U(1)_B \times U(1)_{Qi}$,*

$$\pi_6(SU(5) \times U(1)_B \times U(1)_{Qi}) = 0. \quad (55)$$

For the Euclideanized local Lorentz group $\text{Spin}(6) \simeq SU(4)$,

$$\pi_6(\text{Spin}(6)) = 0. \quad (56)$$

The spin-bordism group (Stong, 1968) satisfies $\Omega_7^{\text{Spin}}(\text{pt}) = 0$. Therefore the GG-6 bulk has no six-dimensional Witten-type global gauge or Lorentz anomaly. This proposition is the sphere-background special case of a stronger statement: the full spin bordism computation of Lemma 6.6 (Subsection 6.3) shows the same conclusion over every closed spin base, not only S^6 .

6.3 Unrestricted global anomaly freedom: the theorem without the topological fine print

What this subsection adds. The theorem just proved is unconditional on the class Admissible, and Admissible is restrictive. Clause (C7) demands $H^1(M_4; \mathbb{Z}) = H^3(M_4; \mathbb{Z}) = 0$; clauses (C6) and (C8) demand that the large gauge transformations checked be null-homotopic, with no Abelian winding and no mixed clutching. Those clauses have earned their keep — they made the bounding construction of Appendix I explicit — but they also left a standing question, named in this paper as Open Problem O10: does the consistency survive when the fine print is removed? A hostile reader is entitled to suspect that the restrictions are where the difficulty was hidden. The present subsection answers the question. The restrictions can be deleted. The global anomaly vanishes for *every* large gauge transformation of the GG-6 bulk, of arbitrary topological type, on *every* compact spin base — and the proof is short enough, and its ingredients pretty enough, that we give every step.

What a global anomaly is, and how the proof so far handled it. An infinitesimal gauge variation is a small deformation; Section 5 showed that the phase it produces is canceled exactly by the Green–Schwarz counterterm. A *large* gauge transformation is a loop: one walks the background around a closed path in configuration space and asks whether the quantum theory comes back to itself with phase one. The standard technology evaluates that phase on the *mapping torus* U_7 — the seven-dimensional manifold obtained by gluing the ends of $(\text{background}) \times [0, 1]$ with the transformation — and, when U_7 together with all its data bounds an eight-manifold W_8 , the Atiyah–Patodi–Singer machinery converts the phase into quantities computed on W_8 . The theorem above walked exactly this path, and clause (C7) entered at a single step: the extension of the gauge data over the filling. The bounding lemma of Appendix I argues that every allowed gauge transformation on the base is *null-homotopic*,

$$[M_4, U(1)] \cong H^1(M_4; \mathbb{Z}) = 0, \quad [M_4, \text{SU}(5)] \cong H^3(M_4; \mathbb{Z}) = 0,$$

so the clutching is trivial and the bundle extends over the product filling $M_4 \times D^2 \times T_{r\theta}^2$. That is clause (C7) at work: it makes the transformations tame enough that the extension problem never arises.

The reframing: from null-homotopy to bordism. Here is the observation that opens the problem. The physics never needed the extension to be *trivial*; it needed *a filling* — any filling — on which the phase evaluates to an integer. Whether the specific product manifold $M_4 \times D^2 \times T_{r\theta}^2$ carries an extension is a question in obstruction theory, and clause (C7) was the answer to that question. Whether *some* spin eight-manifold carries an extension is a different and far more structural question: it asks whether the pair (mapping torus, gauge bundle) represents zero in the spin bordism group of the classifying space,

$$[U_7, P] \in \Omega_7^{\text{spin}}(BG), \quad G = \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}.$$

If that group vanishes, then every closed spin seven-manifold carrying a G -bundle bounds a spin eight-manifold over which the bundle extends — *by the definition of the bordism relation*, with no hypothesis on the topology of the base and no hypothesis on the homotopy class of the transformation. The null-homotopy clause would then stand revealed as a crutch of the construction, not a requirement of the physics. So the task divides in three: compute the group (Lemma 6.6); deal with the one genuinely new danger that abstract fillings introduce, which is their signature (Lemma 6.8); and re-run the four-step evaluation of Theorem 6.2 on the new fillings (Theorem 6.9).

The imported toolbox, stated honestly. Three standard facts are used, each with a one-line statement of what it is doing for us.

- *The spin bordism coefficients.* $\Omega_q^{\text{spin}}(\text{pt})$ for $q = 0, \dots, 7$ is $\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0$ (Milnor; Anderson–Brown–Peterson). The two $\mathbb{Z}_2$ entries in degrees one and two are the eta-invariant layers — the circle with its periodic spin structure and its square — and they are the only rooms in which a torsion global anomaly could live.

- *The Atiyah–Hirzebruch spectral sequence.* $E_{p,q}^2 = H_p(BG; \Omega_q^{\text{spin}}) \Rightarrow \Omega_{p+q}^{\text{spin}}(BG)$: the bordism of a space is assembled from its ordinary homology with bordism coefficients, corrected by differentials.
- *The first differential is a Steenrod square.* In degrees ≤ 7 spin bordism agrees with connective real K-homology, whose first Postnikov invariant is Sq^2 ; consequently the d_2 differentials off the $q = 0$ and $q = 1$ rows are (mod-2 reduction followed by) the dual of Sq^2 . This is the single deep input, and it is what connects the anomaly question to the cohomology operations computed next.

Step 1: the Wu formula, derived from scratch. We need the action of Sq^2 on Chern classes mod 2, and rather than cite it we derive it, because one coefficient in it is a trap. Work by the splitting principle: pull back to a space where the rank-five bundle splits into line bundles with first Chern classes $t_1, \dots, t_5$ (degree two each), so that $c_k = \sigma_k(t_1, \dots, t_5)$, the elementary symmetric polynomial. On a degree-two line class the total Steenrod square is

$$\text{Sq}(t) = t + t^2 \quad (\text{nothing above the top square}), \quad (57)$$

and the total square is multiplicative, so on a monomial $\text{Sq}(\prod_i t_i^{a_i}) = \prod_i (t_i + t_i^2)^{a_i}$. Extracting the degree- $(2k+2)$ part of $\text{Sq}(x^k)$ for a single line class gives the binomial coefficient $\binom{k}{1}$:

$$\text{Sq}^2(x^k) = k x^{k+1} \pmod{2}, \quad (58)$$

so in particular $\text{Sq}^2(x^2) = 0$ and $\text{Sq}^2(x^3) = x^4$: squares are inert, cubes are not. For $\sigma_2 = \sum_{i<j} t_i t_j$, the Cartan formula on each monomial gives $\text{Sq}^2(t_i t_j) = t_i^2 t_j + t_i t_j^2$, hence $\text{Sq}^2 \sigma_2 = \sum_{i<j} (t_i^2 t_j + t_i t_j^2)$. Now compare with the product $\sigma_1 \sigma_2$: multiplying out,

$$\sigma_1 \sigma_2 = \left(\sum_i t_i \right) \left(\sum_{j<k} t_j t_k \right) = \sum_{i<j} (t_i^2 t_j + t_i t_j^2) + 3 \sigma_3,$$

because each square-free product $t_i t_j t_k$ arises from three choices of which factor came from σ_1 . Rearranging mod 2,

$$\text{Sq}^2(c_2) = c_1 c_2 + 3 c_3 = c_1 c_2 + c_3 \pmod{2}. \quad (59)$$

The same computation one degree up: $\text{Sq}^2(t_i t_j t_k) = t_i^2 t_j t_k + t_i t_j^2 t_k + t_i t_j t_k^2$, and $\sigma_1 \sigma_3 = \sum_{i<j<k} (t_i^2 t_j t_k + t_i t_j^2 t_k + t_i t_j t_k^2) + 4 \sigma_4$, whence

$$\text{Sq}^2(c_3) = c_1 c_3 + 4 c_4 = c_1 c_3 \pmod{2} \quad (60)$$

(the Wu formula’s integral coefficient is $2c_4$; the symmetric-function route taken here produces $4\sigma_4$ — the two differ by the even term $2\sigma_4$ and agree mod 2, which is all the argument uses). Here is the trap the derivation exists to avoid: the naive guess “ $\text{Sq}^2 c_k = c_{k+1}$ at $c_1 = 0$ ” is *false* for $k = 3$ — the coefficient of c_4 is four, which is zero mod 2. In the special unitary case ($c_1 = 0$) the upshot is

the asymmetric pair

$$\boxed{\text{Sq}^2 c_2 = c_3, \quad \text{Sq}^2 c_3 = 0} \quad \text{in } H^*(BSU(5); \mathbb{F}_2). \quad (61)$$

The first of these identities will turn out to carry the entire $SU(5)$ side of the global anomaly question.

Step 2: the mod-2 cohomology of BG and the derivation property. The classifying space of the GG-6 gauge group has

$$H^*(BG; \mathbb{F}_2) = \mathbb{F}_2[c_2, c_3, c_4, c_5, x_B, x_{Qi}], \quad \deg c_k = 2k, \quad \deg x = 2,$$

with x_B, x_{Qi} the first Chern classes of the two Abelian factors. Two structural facts do most of the work below. First, *there is no odd-degree cohomology at all*: every class is a polynomial in even generators. Second, the integral cohomology is torsion-free, so Sq^1 (the Bockstein of the mod-2 reduction) vanishes on every class in sight, and the Cartan formula $\text{Sq}^2(uv) = \text{Sq}^2 u \cdot v + \text{Sq}^1 u \text{Sq}^1 v + u \cdot \text{Sq}^2 v$ loses its middle term: Sq^2 acts as a *derivation*. With the generator values (61) and $\text{Sq}^2 x = x^2$, the complete action on the basis elements we need is

$$\begin{aligned} \text{Sq}^2 x_i &= x_i^2, & \text{Sq}^2(x_i^2) &= 0, & \text{Sq}^2(x_1 x_2) &= x_1^2 x_2 + x_1 x_2^2, \\ \text{Sq}^2 c_2 &= c_3, & \text{Sq}^2 c_3 &= 0, & \text{Sq}^2(c_2 x_i) &= c_3 x_i + c_2 x_i^2, \\ \text{Sq}^2(x_i^3) &= x_i^4, & \text{Sq}^2(x_1^2 x_2) &= x_1^2 x_2^2 = \text{Sq}^2(x_1 x_2^2), \end{aligned} \quad (62)$$

where x_1, x_2 abbreviate x_B, x_{Qi} . Every line of (62) is either a generator value or two applications of the Leibniz rule; every line has been verified independently by the splitting-principle computation.

Lemma 6.6 (Vanishing of the spin bordism of the GG-6 gauge group). *For $G = SU(5) \times U(1)_B \times U(1)_{Qi}$,*

$$\tilde{\Omega}_5^{\text{spin}}(BG) = 0, \quad \tilde{\Omega}_7^{\text{spin}}(BG) = 0.$$

Together with $\Omega_5^{\text{spin}}(\text{pt}) = \Omega_7^{\text{spin}}(\text{pt}) = 0$: every closed spin five- or seven-dimensional manifold equipped with a G -bundle bounds a compact spin manifold over which the bundle extends.

Proof. Run the Atiyah–Hirzebruch spectral sequence. Because BG has only even cells, every entry $E_{p,q}^2$ with p odd vanishes; because $\Omega_q^{\text{spin}} = 0$ for $q = 3, 5, 6, 7$, most even- p entries vanish too. What survives in the two total degrees of interest is exactly one entry each:

$$\text{degree 5 : } E_{4,1}^2 = H_4(BG; \mathbb{F}_2) \cong \mathbb{F}_2^4, \quad \text{degree 7 : } E_{6,1}^2 = H_6(BG; \mathbb{F}_2) \cong \mathbb{F}_2^7,$$

with monomial bases (written as cohomology, and read dually) $\{c_2, x_1^2, x_1 x_2, x_2^2\}$ in degree four and $\{c_3, c_2 x_1, c_2 x_2, x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3\}$ in degree six. Both sit on the $q = 1$ row — the first eta layer — so both are subject to the d_2 -differentials dual to Sq^2 , leaving the group and entering it:

$$E_{p+2,0}^2 \xrightarrow{d_2} E_{p,1}^2 \xrightarrow{d_2} E_{p-2,2}^2.$$

We compute both, in both degrees, from the table (62).

Total degree five. The outgoing differential is dual to $\text{Sq}^2 : H^2 \rightarrow H^4$, which sends $x_i \mapsto x_i^2$. Dually, the classes x_1^2 and x_2^2 are mapped onto x_1 and x_2 , while c_2 and x_1x_2 are annihilated (nothing in H^2 hits them):

$$\ker(d_2^{\text{out}}) = \text{span}\{c_2, x_1x_2\}^* \quad (\text{dimension two}).$$

The incoming differential from $E_{6,0}^2$ is dual to $\text{Sq}^2 : H^4 \rightarrow H^6$. From (62), $\text{Sq}^2c_2 = c_3$ and $\text{Sq}^2(x_1x_2) = x_1^2x_2 + x_1x_2^2$; dually, the degree-six classes c_3 and $x_1^2x_2 + x_1x_2^2$ map precisely onto c_2 and x_1x_2 . (The mod-2 reduction $H_6(BG; \mathbb{Z}) \rightarrow H_6(BG; \mathbb{F}_2)$ is onto, the integral cohomology being torsion-free.) So

$$\text{im}(d_2^{\text{in}}) = \text{span}\{c_2, x_1x_2\}^* = \ker(d_2^{\text{out}}),$$

kernel equals image, both of dimension two, and $E_{4,1}^3 = 0$.

Total degree seven. The outgoing differential is dual to $\text{Sq}^2 : H^4 \rightarrow H^6$, just computed: it kills the dual of c_3 (because $\text{Sq}^2c_2 = c_3$) and identifies the duals of $x_1^2x_2$ and $x_1x_2^2$ pairwise (because $\text{Sq}^2(x_1x_2)$ is their sum), so

$$\ker(d_2^{\text{out}}) = \text{span}\{c_2x_1, c_2x_2, x_1^3, x_2^3, x_1^2x_2 + x_1x_2^2\}^* \quad (\text{dimension five}).$$

The incoming differential from $E_{8,0}^2$ is dual to $\text{Sq}^2 : H^6 \rightarrow H^8$. Read the last two lines of (62): the five degree-eight classes $c_3x_1, c_3x_2, x_1^4, x_2^4, x_1^2x_2^2$ pull back, dually, onto exactly the five kernel elements — c_3x_i hits c_2x_i (from $\text{Sq}^2(c_2x_i) = c_3x_i + c_2x_i^2$), x_i^4 hits x_i^3 (from $\text{Sq}^2x_i^3 = x_i^4$), and $x_1^2x_2^2$ hits the sum $x_1^2x_2 + x_1x_2^2$ (both preimages map to it). Note what does *not* happen: $\text{Sq}^2c_3 = 0$, so nothing arrives at the dual of c_3 from above — but nothing needs to, because that class already died leaving. So again

$$\text{im}(d_2^{\text{in}}) = \ker(d_2^{\text{out}}),$$

kernel equals image, both of dimension five, and $E_{6,1}^3 = 0$. In both total degrees every entry is now zero, so no higher differential has a source or a target, no extension problem arises, and both reduced bordism groups vanish. ■

Remark 6.7 (The two mechanisms, seen separately). It is instructive to run the two factors of the gauge group alone. For $\text{SU}(5)$ by itself the single degree-seven candidate is the dual of c_3 , and it dies *leaving*: the outgoing differential is injective because $\text{Sq}^2c_2 = c_3$. For a single $U(1)$ the candidate is the dual of x^3 , and it dies *arriving*: the incoming differential is surjective because $\text{Sq}^2x^3 = x^4$. Two different doors, one conclusion. The proposition of Section 5 on Witten-type anomalies — $\pi_6(G) = 0$ — is the sphere-background special case of Lemma 6.6: the homotopy groups see the anomaly candidates over S^6 , the bordism group sees them over every closed spin base at once.

Lemma 6.8 (Signature normalization of fillings). *If a closed spin seven-manifold with G -bundle (U_7, P) bounds $(W_8, \tilde{P})$, then it bounds a spin eight-manifold of signature zero carrying the same boundary data.*

Proof. The quaternionic projective plane $\mathbb{HP}^2$ is 3-connected, hence spin, and has signature one (its middle cohomology is one-dimensional, generated by u with $\int u^2 = 1$). Form the interior connected

sum $W' = W \# k(\mp \mathbb{H}\mathbb{P}^2)$ with $k = |\sigma(W)|$ and orientations chosen to cancel: the sum is taken along balls in the interior, so the boundary is untouched; the spin structures glue because both sides are spin and the gluing sphere is simply-connected; and the G -bundle extends trivially over each punctured $\mathbb{H}\mathbb{P}^2$ summand because a bundle over a ball is trivial. Then $\sigma(W') = \sigma(W) \mp k = 0$ by Novikov additivity, and $\partial W' = U_7$ with unchanged data. ■

Why the signature needed a lemma at all. On the product fillings of Theorem 6.2 the relative intersection form vanishes identically and the self-dual tensor sector's signature-type phase $\exp(2\pi i n_{\text{SD}} \sigma(W)/8)$ never had to be confronted. On an abstract filling it would be — and at $n_{\text{SD}} = 4\Omega = 100$ the exponent is $2\pi i \cdot 12.5\sigma$, genuinely fractional for odd σ . The lemma says the danger is illusory because *signature is purchasable*: a closed spin eight-manifold of signature one exists, so the signature of a filling can always be paid down to zero, at zero cost to anything the boundary sees. The product structure of the original fillings was quietly doing this job; surgery does it in general.

Theorem 6.9 (T-BI6-7.6: Unrestricted global anomaly freedom). *Let Admissible^+ denote the class obtained from Definition 6.1 by deleting clause (C7) and deleting the topological restrictions on large transformations from clauses (C6) and (C8) — retaining only structure preservation (transformations preserve the spin structure and the stage data). Then for every background in Admissible^+ and every large gauge transformation of arbitrary topological type — $\text{SU}(5)$ clutching classes $\delta(g) \in H^3(M_4; \mathbb{Z})$, Abelian winding classes in $H^1(M_4; \mathbb{Z})$, mixed clutching in $H^2(M_4; \mathbb{Z}) \otimes H^1(T_{r\theta}^2; \mathbb{Z})$ — the combined phase of the chiral determinant and the Wu–Chern–Simons-refined BI-6 counterterm around the mapping torus is trivial. More generally, the phase is trivial on every closed spin seven-manifold equipped with a G -bundle, within the differential-cohomology accounting of the present Part. The statement holds at every point $\gamma = s/t$ of the coefficient family of the BI-6 clause, for any chiral spectrum achieving the factorization at that γ (the exhibited spectrum of Theorem 7.9 is the $\gamma = 2/5$ witness).*

Proof. Four steps, of which only the first is new; the remaining three are the machinery already established, now applied without restriction.

Step 1 (bounding). The mapping-torus datum (U_7, P) is a closed spin seven-manifold with G -bundle — whatever the topological type of the transformation that produced it. By Lemma 6.6 it bounds $(W_8, \tilde{P})$; the connections and the differential cocycle $\check{X}$ extend over W_8 exactly as in the bounding lemma of Appendix I (extend the connection arbitrarily; build the refinement from the extended data).

Step 2 (normalization). By Lemma 6.8, arrange $\sigma(W_8) = 0$.

Step 3 (evaluation). Run the four-part accounting of Theorem 6.2 on W_8 . The fermionic contributions are Atiyah–Patodi–Singer indices of the extended operators — integers. The signature-type phase of the self-dual tensor sector is $\exp(2\pi i n_{\text{SD}} \cdot 0/8) = 1$. The Wu–Chern–Simons term is an integer by the quantization lemma, which holds on *any* spin eight-manifold — its own statement requires nothing beyond spin plus integrality. The integral lift exists by the coefficient-family lift lemma, which holds on any background. The local index densities cancel pointwise against the

counterterm curvature by the factorization assumed at γ — Theorem 7.9 witnesses its $\gamma = 2/5$ point. The total exponent is an integer; the phase is one.

Step 4 (filling-independence). Two signature-zero fillings W, W' glue along U_7 to a closed spin eight-manifold with $\sigma = \sigma(W) - \sigma(W') = 0$ by Novikov additivity, on which the same accounting yields an integer; so the value computed in Step 3 does not depend on the choices made in Steps 1 and 2. ■

Remark 6.10 (What the theorem does not need). It does not need M_4 to have vanishing H^1 or H^3 . It does not need the gauge transformations to be null-homotopic, unwound, or unmixed. It does not need the bundle to be pulled back from the base. And — a bonus that fell out of the generality — it does not even need the transformation to be a gauge transformation: a large diffeomorphism preserving the spin structure produces a mapping torus that is still a closed spin seven-manifold with G -bundle, and Steps 1–4 never ask for more. The clauses of Definition 6.1 that restricted the transformations survive as a description of the physical intended backgrounds — the stage on which the cohort’s phenomenology is computed — but they no longer bear the weight of the consistency theorem.

Remark 6.11 (Three observations on why it works). *First: the cubic Casimir does double duty.* Theorem 5.3 demanded, as a rational statement, that the cubic index of the charged matter vanish — the condition that fixed $L = 5$ in Theorem 9.2. The bordism computation now meets the same class one level down: as a mod-2 class, c_3 is $\text{Sq}^2 c_2$, and it is precisely this identity that makes the outgoing differential injective and kills the last candidate $\mathbb{Z}_2$ global anomaly of the $\text{SU}(5)$ sector. The cubic anomaly’s rational part selects the group; its mod-2 shadow certifies the group globally safe. We record this as a resonance, not a mechanism — but it is the kind of resonance that suggests the bookkeeping is being kept in the right variables. *Second: a gauge group with an even classifying space cannot hide a global anomaly for long.* Torsion global anomalies in seven dimensions must live on odd-degree bordism classes; BG has no odd cells, so the only candidates enter through the two eta layers of the spin coefficients, and those are exactly the layers the first k-invariant of real K-theory — Sq^2 — can reach. In the GG-6 case it reaches all of them. The constitution selected a gauge group whose topology leaves the anomaly nowhere to sit. *Third: signature is purchasable.* The one quantity that distinguished the tame fillings from the wild ones can be adjusted by surgery at no physical cost. What looked like a wall was a ledger entry.

Remark 6.12 (Scope, stated at the same standard as the rest of the Part). Theorem 6.9 is proved within the differential-cohomology phase accounting that this Part uses throughout — the same standard to which Theorem 6.2 itself is proved on the intended class. The foundation of that accounting for the non-supersymmetric self-dual sector is supplied in Subsection 6.4: the defining quadratic refinement is proved unique (Theorem 6.13) and the partition function is constructed (Theorem 6.18), so what formerly stood beneath both theorems as Open Problem O9 now stands as a theorem beneath them. Within the accounting, Open Problem O10 — the extension of the global theorem beyond the intended class — is discharged for the GG-6 gauge content: what remains genuinely open in O10’s original wording is the case of *arbitrary* gauge packets and tensor lattices, which is the six-dimensional anomaly community’s general classification program rather than a gap in the GG-6 chain.

6.4 The self-dual partition function: the refinement is unique, and the construction owes nothing to supersymmetry

What is still missing. Everything proved so far about the tensor sector — the Wu–Chern–Simons quantization, the signature accounting, the global theorem — concerns the *anomaly* of the self-dual field: the seven-dimensional invertible theory that governs its phases. Open Problem O9 has, until now, recorded what was missing underneath: the six-dimensional partition function of the self-dual sector itself. That object is not defined by a Lagrangian; in the framework this Part uses, it is defined through a *quadratic refinement* — and the only explicit construction in the literature, due to Monnier and Moore, is carried out for $\mathcal{N} = (1, 0)$ supergravity, whose supersymmetry is entangled with the construction at several points. The worry is easy to state and serious: if the non-supersymmetric BI-6 sector required a *choice* of refinement that nothing in the theory fixes, then a hidden discrete dial would sit beneath the central consistency theorem of this paper. The present subsection shows the worry is empty, in the strongest available sense: on the BI-6 tensor lattices the refinement admits *no choice at all*, every ingredient of the construction that supersymmetry supplies in the $(1, 0)$ case is supplied here by the constitution or by a theorem already proved, and the partition function can then simply be written down.

Step 1: why the self-dual field has no Lagrangian. It is worth deriving this rather than asserting it, because it explains why the entire subsection must run through period lattices rather than path integrals. Take the free action of an unconstrained two-form, $S = \frac{1}{2} \int_{M_6} H_3 \wedge *H_3$ with $H_3 = dB_2$. Its equation of motion is $d * H_3 = 0$, which together with the Bianchi identity $dH_3 = 0$ describes a two-form with *both* duality components propagating — twice the content wanted. Suppose instead one imposes self-duality $H_3 = *H_3$ directly in the action. Then

$$S = \frac{1}{2} \int_{M_6} H_3 \wedge H_3 = \frac{1}{2} \int_{M_6} d(B_2 \wedge dB_2) = 0 \quad (63)$$

on a closed manifold: the would-be action is a total derivative and vanishes identically. Self-duality cannot be imposed by any local covariant action for B_2 alone; the constraint kills the very term that would enforce it. The self-dual field is, in this precise sense, *half* of an ordinary two-form — the six-dimensional sibling of the chiral boson in two dimensions — and one must quantize the whole and then take the half at the level of the partition function.

Step 2: the accepted definition — holomorphic factorization. The construction, standard since the five-brane analyzes of the 1990s and formalized in the Hopkins–Singer / Monnier–Moore framework, runs as follows. Quantize the *non-chiral* two-form, which does have a Lagrangian; its partition function factorizes into a sum of products of holomorphic and antiholomorphic *conformal blocks*; the self-dual field’s partition function is declared to be a single block. Where do the blocks live? The fluxes of H_3 take values in the lattice $H^3(M_6; \mathbb{Z})$, and the zero-mode sector of the path integral is a Gaussian sum over this lattice — a theta function on the *intermediate Jacobian*

$$\mathcal{J}^3(M_6) = H^3(M_6; \mathbb{R}) / H^3(M_6; \mathbb{Z}). \quad (64)$$

Two pieces of structure make this torus a natural home for theta functions, and both are worth deriving on the spot. First, the *complex structure*: in Euclidean signature the Hodge star on k -forms in n dimensions satisfies $*^2 = (-1)^{k(n-k)}$, and for three-forms in six dimensions $k(n-k) = 9$, so

$$*^2 = -1 \quad \text{on three-forms:} \quad (65)$$

the metric's Hodge star *is* a complex structure on $H^3(M_6; \mathbb{R})$. (This is also why the natural language of the Euclidean self-dual field is complex; in Lorentzian signature $*^2 = +1$ and the chiral split is real.) Second, the *polarization*: the intersection pairing $\langle x, y \rangle = \int_{M_6} x \wedge y$ on degree-three classes is antisymmetric ($x \wedge y = (-1)^{3 \cdot 3} y \wedge x$) and unimodular by Poincaré duality — a symplectic lattice, which forces $b_3(M_6)$ to be even. A torus with a complex structure and a principal symplectic polarization is precisely a principally polarized abelian variety, and its natural functions are theta functions.

Step 3: choosing a block is choosing a quadratic refinement. A theta function on a principally polarized torus is not unique: it carries a *characteristic*, and the choice of characteristic is exactly the choice of conformal block. The cleanest way to see what that choice means physically is the rank-one computation. The classical Jacobi function with even characteristic obeys

$$\theta_3(z + \pi | \tau) = \theta_3(z | \tau), \quad \theta_3(z + \pi\tau | \tau) = e^{-i\pi\tau - 2iz} \theta_3(z | \tau) : \quad (66)$$

under a shift by a lattice vector the function returns to itself up to a multiplier $e^{-i\pi\tau - 2iz}$ whose exponent is *quadratic* in the shift — iterating the second identity m times produces $e^{-i\pi m^2 \tau - 2imz}$. The multiplier system of a theta function is the exponential of a quadratic form on the lattice, and consistency of the multipliers under composition of shifts is precisely the condition

$$q(x + y) - q(x) - q(y) = \langle x, y \rangle \pmod{1} : \quad (67)$$

q is a *quadratic refinement* of the pairing. The characteristic of the theta function and the refinement are two names for the same discrete datum. By contrast the odd-characteristic function θ_1 satisfies $\theta_1(z + \pi) = -\theta_1(z)$: a block with the wrong characteristic *changes sign* under a lattice shift. Hold that sign; it returns in Step 5 as the executioner of the wrong choices.

Step 4: the refinements the Wu lemma admits, and their uniqueness. On the Green–Schwarz side the refinement appears in the form this Part has used all along:

$$q_c(x) = \frac{1}{2}(x \cdot x - c \cdot x), \quad x \in \Lambda, \quad (68)$$

on the tensor charge lattice Λ , where c is a *characteristic element*:

$$x \cdot x \equiv c \cdot x \pmod{2} \quad \text{for all } x \in \Lambda. \quad (69)$$

Condition (69) is exactly what makes q_c integer-valued, which is what the Wu–Chern–Simons quantization lemma consumes; it is the lattice-side twin of the spacetime statement that $\lambda = p_1/2$ is a characteristic element of the intersection form on spin eight-manifolds. So the question “which refinement?” becomes: *how many characteristic elements does the BI-6 lattice have, up to equivalence?*

Theorem 6.13 (T-BI6-7.7: Uniqueness of the Wu–Chern–Simons refinement). *Let Λ be any unimodular integral lattice. Its characteristic elements form a single coset of 2Λ , and the Wu–Chern–Simons functional (278) depends on the characteristic element only through the curvature of the shift: two elements c and $c + 2u$ give functionals on a closed seven-manifold that differ by $\int \check{X} \cup (u \otimes \check{\lambda})$, whose curvature $u X_4 \wedge \lambda$ is a term of the anomaly polynomial. The choice is therefore fixed by the curvature match with I_8 and is not a discrete datum. In particular, for the BI-6 tensor sector — the rank-one mediator lattice $\Lambda \cong \mathbb{Z}$ of clause (C3), on which the characteristic elements are exactly the odd integers and the curvature match selects $a = 1$, together with the diagonal unimodular spectator lattice $\mathbb{Z}^{4\Omega-1}$, on which the X_4 -source vanishes and the shift is empty — the Wu–Chern–Simons refinement is **unique**. There is no discrete choice to make.*

Remark 6.14 (What the uniqueness spares, read against a four-dimensional completion). The uniqueness is what makes the OGN-5 ledger compulsory rather than optional. A four-dimensional-only completion of gauged baryon number — the construction of (Benakli, Goodsell, and Goudeau, 2026) is the concrete case, engaged in Subsection 11.6 — carries no tensor sector, so nothing in it plays the role the present theorem plays here. Embedded as the boundary of a six-dimensional bulk, it would need the rank-one lattice, its unimodularity, the characteristic element, and the coefficient vector each as an independent axiom, and would still meet $\pi_6(\mathrm{SU}(3)) = \mathbb{Z}_6$ and $\pi_6(\mathrm{SU}(2)) = \mathbb{Z}_{12}$ (Mimura and Toda, 1963, Toda, 1962), where $\mathrm{SU}(5)$ gives zero in the stable range.

Proof. Single coset. Let c, c' both satisfy (69). Then $(c - c') \cdot x \equiv 0 \pmod{2}$ for every x , so $\frac{1}{2}(c - c')$ pairs integrally with all of Λ , i.e. $\frac{1}{2}(c - c') \in \Lambda^* = \Lambda$ by unimodularity. Hence $c - c' \in 2\Lambda$.

The shift, and what fixes it. Write $c' = c + 2u$ with $u \in \Lambda$. On the lattice,

$$q_{c'}(x) = \frac{1}{2}(x \cdot x - c \cdot x) - u \cdot x = q_c(x) - u \cdot x, \quad (70)$$

and since every q_c is integer-valued by (69), the two lattice refinements cannot be told apart modulo 1 — a comparison that is empty rather than a proof, because it would hold for any pair of integer-valued functions. The object that has to be unique is the $\mathbb{R}/\mathbb{Z}$ -valued functional (278), in which the characteristic element enters through the Wu class $\check{\lambda}$: replacing c by $c + 2u$ shifts S_{WCS} by $\int_{W_8} \check{X} \cup (u \otimes \check{\lambda})$, a term whose curvature is $u X_4 \wedge \lambda$. That is not an invisible ambiguity; it is a term of the anomaly polynomial, and the anomaly polynomial is fixed by the spectrum. The curvature match of $\frac{1}{2}\check{X} \cup \check{X} - \frac{1}{2}\check{X} \cup \check{\lambda}$ with I_8 therefore selects the element, and a second characteristic element would change the polynomial the counterterm cancels.

The BI-6 lattices. On $\Lambda = \mathbb{Z}$ with $x \cdot y = xy$, condition (69) reads $x^2 \equiv cx \pmod{2}$ for all x ; taking $x = 1$ forces c odd, and c odd suffices since $x^2 \equiv x \pmod{2}$ identically. The odd integers are the coset $1 + 2\mathbb{Z}$, and the curvature match with I_8 — the tensor’s own $\frac{1}{2}X_4^2$ against the spectrum’s

— selects $a = 1$, the very value clause (C3) carries. On the spectator lattice the same argument runs componentwise: characteristic vectors are the all-odd vectors, the coset $(1, \dots, 1) + 2\mathbb{Z}^{4\Omega-1}$; the spectators are X_4 -decoupled, so their $\check{X}$ -source vanishes and the shift $\int \check{X} \cup (u \otimes \check{\lambda})$ is empty, whichever all-odd vector is taken. ■

Remark 6.15 (One refinement, three appearances). The minimal characteristic element $a = 1$ fixed here is the same integer quadratic form $q(x) = \frac{1}{2}(x^2 - x) = x(x-1)/2$ that prices the family cascade of GG-P2-N3 (Arisaka, 2026, P2-N3) — its toll ladder satisfies the refinement’s defining functional equation exactly — and whose Hopkins–Singer pushforward that paper’s Appendix H identifies, on the benchmark background of the intended class of this section, with the boundary’s Berry line bundle (the κ -bridge theorem). One refinement, seen three ways: the tensor sector, the toll ledger, and the seam.

Remark 6.16 (What supersymmetry was actually doing, located exactly). Theorem 6.13 holds for every unimodular lattice — also for the signature- $(1, n_T)$ string-charge lattices of $(1, 0)$ supergravity. Uniqueness of the refinement was therefore never the supersymmetric input. What supersymmetry supplies in the $(1, 0)$ construction is something logically prior: the theorem (due to the six-dimensional supergravity literature) that the *gravitational anomaly vector* a is itself a characteristic element of Λ , i.e. that $x \cdot x + a \cdot x \in 2\mathbb{Z}$ — without which the Wu–Chern–Simons theory does not exist at all. That is a genuine theorem there, derived from the supersymmetric anomaly-factorization conditions. In the BI-6 setting the corresponding statement is one line of arithmetic: $a = 1$ on $\Lambda = \mathbb{Z}$, and 1 is characteristic because $x^2 \equiv x \pmod{2}$. The existence condition that costs supersymmetry a theorem costs the present paper nothing — clause (C3), written to normalize the Green–Schwarz coupling, had already recorded the answer.

Step 5: the residual twists, and their double execution. One tier above the lattice, two discrete deformations could still be *added* as background data, and honesty requires naming and killing both. A *flux character* shifts the refinement by a linear form, $q(x) \mapsto q(x) + u \cdot x$ with u a fixed half-lattice or torsion class — a discrete theta angle. A *Wu-structure twist* replaces the canonical integral Wu lift $\lambda = p_1/2$ that the spin structure provides by another lift, shifting the refinement by a mod-2 character. Both die twice over.

Lemma 6.17 (Exclusion of refinement twists). *On the BI-6 tensor lattices, every deformation of the refinement away from the canonical class of Theorem 6.13 either fails integer-valuedness — and with it the Wu–Chern–Simons quantization lemma — or renders the partition function double-valued. Independently, any such deformation is unforced discrete background structure excluded by AX-4 (MSMF) and POS-6 (OGN).*

Proof. The minimal flux twist on the mediator replaces $q_1(x) = (x^2 - x)/2$ by $q'(x) = q_1(x) + \frac{x}{2} = (x^2 - 0 \cdot x)/2$, whose “characteristic” is even and therefore not characteristic: at $x = 1$, $q'(1) = \frac{1}{2} \notin \mathbb{Z}$, and the integrality on which the quantization lemma rests is gone. At the level of the theta block the same twist moves the characteristic from even to odd, and by the shift behavior recorded in Step 3 the block then obeys $\theta_1(z + \pi) = -\theta_1(z)$: the would-be partition function returns to *minus* itself under a lattice identification, i.e. it is not a function on the background space at all. A Wu twist

acts identically through its induced character. This is the structural execution. The constitutional one is independent: a flux character or Wu twist is a new discrete background field, forced by no clause, generated by no bulk field, and absent from the OGN-5 ledger; AX-4 excludes it as unforced structure exactly as it excludes unforced tensor multiplication (Theorem 5.1) and unforced Abelian asymmetry ($c = d$), and POS-6 excludes it as normalization freedom outside the ledger. ■

Step 6: the supersymmetry audit. With the refinement question settled, what remains of O9’s worry is the suspicion that the $(1, 0)$ construction uses supersymmetry somewhere else essential. It does not, and the accounting is short enough to print. Table 4 lists each ingredient of the Monnier–Moore construction with what supplied it there and what supplies it here.

Table 4: Each ingredient of the Monnier–Moore construction, with what supplies it in $(1, 0)$ supergravity and what supplies it here. The table answers one specific objection — that the construction secretly uses supersymmetry somewhere — and it answers it by enumeration rather than by assertion. Read the third column: every ingredient is supplied either by a named constitutional clause or by a theorem of this paper, and none by supersymmetry. The accounting is short enough to print in full, which is why it is printed rather than summarized.

Ingredient	In $(1, 0)$ supergravity	In GG-6/BI-6
Unimodular tensor lattice Λ	string-charge completeness arguments	clause (C3): $\Lambda \cong \mathbb{Z}$, rank one, primitive — AX-4/POS-6
Characteristic property of the gravitational vector a	a theorem of the supersymmetric anomaly-factorization conditions	one line of arithmetic: $a = 1$ odd on $\mathbb{Z}$ (Remark 6.16)
Wu structure on spacetime	spin structure of the SUSY multiplet	spin structure of clause (C1); canonical lift $\lambda = p_1/2$
Factorized anomaly polynomial	fixed by the SUSY multiplet content	Theorem 7.9: the benchmark spectrum, by anomaly closure
Fermionic η -invariants pairing the tensor phase	gravitino and tensorino content	the benchmark fermions — different origin, same Atiyah–Patodi–Singer mathematics
Bordism-invariance of the total phase	verified on supersymmetric backgrounds	Theorem 6.9, unrestricted — stronger than the supergravity analogue

Supersymmetry enters the $(1, 0)$ construction through *which fields exist* and *what their coefficients are* — never through the differential-cohomology machinery itself. Both roles are discharged here by the constitution and by theorems already proved. The construction transfers whole.

Step 7: the partition function, written down.

Theorem 6.18 (T-BI6-7.8: The self-dual partition function of the BI-6 tensor sector). *On every background of the (enlarged) admissible class, the partition function of the full tensor sector — the X_4 -coupled mediator together with its $4\Omega - 1$ gauge-singlet spectators — exists, is canonical, and is*

given by

$$Z_{\text{tensor}}(M_6; \check{X}) = \Delta(g) \cdot \Theta_{\text{can}}(\mathcal{J}^3(M_6); \tau(g), \check{X}) \cdot \Theta_{\text{can}}(\mathcal{J}^3(M_6); \tau(g), 0)^{4\Omega-1}, \quad (71)$$

where $\Delta(g)$ is the metric determinant factor of the non-chiral quantization (spectral-zeta regularized; it exists unconditionally and carries the local curvature anomaly that the Green–Schwarz term cancels); Θ_{can} is the theta function on the intermediate Jacobian with modular parameter $\tau(g)$ determined by the Hodge star and with the canonical characteristic — the unique refinement of Theorem 6.13; the mediator’s theta is sourced through the differential cocycle $\check{X}$ (the coupling $\int B_2 \wedge X_4$ shifts the theta argument by the period vector of $\hat{X}$), and the spectators, being X_4 -decoupled, contribute unsourced canonical factors. Z_{tensor} is single-valued over the space of backgrounds and under all large gauge transformations and large diffeomorphisms.

Proof. Existence and canonicity: the block decomposition exists by holomorphic factorization of the non-chiral field (Step 2); the block is pinned by Theorem 6.13 and Lemma 6.17 — there is exactly one admissible characteristic, so “a block” is “the block.” Single-valuedness: transporting Z_{tensor} around a closed loop in the space of backgrounds — a large gauge transformation or large diffeomorphism — multiplies it by the holonomy of the anomaly line over that loop, evaluated on the loop’s mapping torus; by Theorem 6.9 that holonomy is trivial on every closed spin seven-datum. The two theorems are converses in spirit: the unrestricted global theorem proved the anomaly line bundle trivializable; the present construction exhibits the section. ■

Remark 6.19 (The intended class, fully explicit). On the intended backgrounds the abstract objects become concrete. With $H^1(M_4; \mathbb{Z}) = H^3(M_4; \mathbb{Z}) = 0$, the Künneth theorem gives

$$H^3(M_4 \times T_{r\theta}^2; \mathbb{Z}) = H^2(M_4; \mathbb{Z}) \otimes H^1(T_{r\theta}^2; \mathbb{Z}), \quad \text{rank} = 2b_2(M_4), \quad (72)$$

the two H^1 generators of the structural torus pairing the b_2 classes of the base into a symplectic lattice of rank $2b_2$ — even, as Step 2 requires. The intermediate Jacobian is the torus $T^{2b_2(M_4)}$ and Θ_{can} is an ordinary Riemann theta function of genus $b_2(M_4)$ with zero characteristic. For a base with $b_2 = 0$ the Jacobian is a point and Z_{tensor} reduces to the bare determinant factor times the flux-free theta constants.

Remark 6.20 (Two observations, and the honest boundary). *The dial that was never there.* The fear behind O9 was a hidden discrete choice beneath the central theorem — a datum the supersymmetric treatment fixed for its own reasons and the non-supersymmetric theory would have to fix by hand. The arithmetic says the dial does not exist: a unimodular lattice carries exactly one Wu–Chern–Simons refinement, and the rank-one clause of the constitution — written to enforce minimality — also purchased uniqueness of the quantum definition. Minimal structure has no room for moduli; that is AX-4 operating one level deeper than it was aimed. *a = 1 was already the answer.* Clause (C3)’s “characteristic vector $a = 1$ ” was placed there to normalize the Green–Schwarz coupling; it is also, verbatim, the characteristic element that defines the unique refinement. The paper answered O9 in its own definitions before the question was posed sharply. *The honest boundary.* The holomorphic-factorization definition of the self-dual field is the accepted definition throughout

the field — it is the one used by the five-brane literature and by the $(1, 0)$ supergravity treatments alike — and the present subsection adopts it on the same terms. That convention sits upstream of O9 here exactly as it sits upstream of the supersymmetric constructions; nothing in this paper deepens or weakens it. And the numerical evaluation of η -invariants on individual mapping tori, which O9 originally requested as an independent confirmation, is hereby downgraded rather than discharged: every quantity it would compute is now fixed exactly by theorems, so it can catch errors but cannot carry information. It retains the standing of re-running a verified computation.

6.5 Why global anomaly freedom is critical for OGGEF and GDOS

The role of global anomaly freedom becomes even clearer once GG-6 is connected to GDOS. OGGEF is more than a symbolic map from a six-dimensional bulk to a four-dimensional boundary. It is meant to define a lawful, gauge-invariant, single-valued projection channel from bulk data to boundary observables. If the bulk partition function were only locally consistent but globally multi-valued, then the projected boundary dynamics would inherit an ambiguity: two bulk paths related by a large gauge transformation or large diffeomorphism could project to boundary descriptions that differ by an unphysical phase. That would destroy the functorial meaning of the projection and would make the boundary channel depend on topological representatives rather than only on physical data.

Global anomaly freedom removes exactly that danger. It guarantees that the bulk functional integral is a gauge-invariant object on the intended category, so the OGGEF projection does not depend on arbitrary clutching choices, non-contractible loops in background space, or the self-dual tensor's hidden quadratic phase. Operationally, this is the difference between a boundary theory that is merely *locally suggestive* and a boundary theory that is *globally lawful*. Since GDOS treats the observed four-dimensional world as an open-system projection of the six-dimensional bulk, the lawfulness of the bulk is prior to the lawfulness of the boundary channel.

There is also a deeper physics intuition. In ordinary local anomaly cancellation one checks that the equations of motion and Ward identities survive infinitesimal variations. In a globally anomalous theory, however, the very notion of transporting a state or partition function around the allowed space of backgrounds breaks down. OGGEF and GDOS require more than infinitesimal control: they require that the 6D parent theory define a globally consistent packet whose projection to 4D is independent of the route by which one arrived at a given boundary slice. The global theorem is therefore a prerequisite for treating OGGEF as a physically meaningful projection principle.

Lemma 6.21 (Phase propagation through the OGGEF channel). *Let $Z_{\text{bulk}}[\mathcal{A}]$ be the exponentiated bulk functional on the intended GG-6 category, and let*

$$\mathcal{D}_{\text{OGGEF}} : Z_{\text{bulk}}[\mathcal{A}] \mapsto Z_{\text{bdy}}[\mathcal{J}]$$

denote the linear bulk-to-boundary source map used to define the OGGEF generating functional. If under an allowed large transformation g one had

$$Z_{\text{bulk}}[\mathcal{A}^g] = e^{i\phi(g; \mathcal{A})} Z_{\text{bulk}}[\mathcal{A}],$$

then linearity gives

$$Z_{\text{bdy}}[\mathcal{J}]^g = \mathcal{D}_{\text{OGGEP}}(Z_{\text{bulk}}[\mathcal{A}^g]) = e^{i\phi(g;\mathcal{A})} \mathcal{D}_{\text{OGGEP}}(Z_{\text{bulk}}[\mathcal{A}]) = e^{i\phi(g;\mathcal{A})} Z_{\text{bdy}}[\mathcal{J}].$$

Therefore a nontrivial bulk global anomaly phase descends directly to the boundary source functional. In particular, the *OGGEP* channel fails to descend to gauge-equivalence classes unless $\phi(g;\mathcal{A}) \in 2\pi\mathbb{Z}$ for every allowed g .

Proof. The point is linearity, not detailed dynamics. *OGGEP* is read here at the level of amplitudes / generating functionals, upstream of the normalized density-matrix channel $\mathcal{D}_{\text{OGGEP}} = \mathcal{R} \circ \text{Tr}_{\text{env}}$ constructed in (Arisaka, 2026, A5-GDOS); the two readings are compatible because that channel is defined on gauge-equivalence classes of sources, which is precisely what a representative-dependent phase obstructs. A multiplicative phase in the parent functional therefore survives as the same multiplicative phase after projection. Normalized density matrices may wash out a single overall phase on one connected patch, but the channel as a law between gauge-equivalence classes is already ill-defined once the source functional depends on the representative. Hence global anomaly freedom is a prerequisite for a lawful *OGGEP* map. ■

6.6 Constitutional status of $U(1)_{Q_i}$: optional under the constitution, preferred by the empirical data

This subsection is the home of the constitutional admissibility proof for the fourth gauge factor $U(1)_{Q_i}$ of the GG-6 bulk. The anomaly theorem proved in Sections 5–6 goes through whether or not $U(1)_{Q_i}$ is present; the question this subsection settles is what the GG-Theory Constitution itself *requires*, and what (if anything) it merely *permits*. The companion paper (Arisaka, 2026, A2-GG6) cross-cites the present subsection at the points where it adopts the four-component bulk and is explicit that the choice is empirical rather than constitutional; here we give the rigorous statement.

Theorem 6.22 (T-BI6-7.5: Constitutional admissibility of both three-component and four-component bulks). *The four constitutional axioms AX-1–AX-4 and the seven postulates POS-1–POS-7 of (Arisaka, 2026, A1-Constitution) admit two distinct anomaly-free six-dimensional bulk theories:*

$$\begin{aligned} (\text{three-component bulk}) \quad & G_{\text{Bulk}}^{(3)} = \text{Spin}(1, 5) \times \text{SU}(5) \times U(1)_B, \\ (\text{four-component bulk}) \quad & G_{\text{Bulk}}^{(4)} = \text{Spin}(1, 5) \times \text{SU}(5) \times U(1)_B \times U(1)_{Q_i}. \end{aligned} \tag{73}$$

Both versions support a single-tensor Green–Schwarz factorization $I_8 = \frac{1}{2}X_4^2$ with X_4 given by the appropriate truncation of (6), both admit a chiral fermion spectrum saturating the relevant trace-ratio identity, and both close under the *OGN* integer ledger at $L = 5$. The choice between the two bulks is therefore empirical, not constitutional.

Why the constitution does not force $U(1)_{Q_i}$. Indeed, the textual chain “AX-3 + POS-4 + POS-1 $\Rightarrow$ forces $U(1)_{Q_i}$ ” that occasionally appears in early drafts of the GG-Theory program

overstates what the constitutional clauses actually demand. CPTP is a *structural* property of the GKLS channel (Gorini, Kossakowski, and Sudarshan, 1976, Lindblad, 1976) that POS-4 imposes on the bulk-to-boundary registration map; it is not a covariance demand under a fresh $U(1)$. Operational closure (POS-2) and gauge invariance under the existing bulk gauge content (POS-1) together produce a CPTP bulk-to-boundary channel whether or not a fourth gauge factor is present. The conserved information current that the channel registers can be identified either with a slot inside the existing $SU(5) \times U(1)_B$ content (under the three-component reading) or with a separate $U(1)_{Q_i}$ generator (under the four-component reading). Both readings are admissible; neither is forced by the constitutional clauses alone. The substantive axiomatic content is *admissibility* of $U(1)_{Q_i}$ as an extension that does not violate any axiom or postulate; the substantive empirical content is *preference* for the four-component reading on data-driven grounds.

Proof of Theorem 6.22. Anomaly freedom of the three-component bulk $G_{\text{Bulk}}^{(3)} = \text{Spin}(1, 5) \times SU(5) \times U(1)_B$ is the truncation $F_{Q_i} = 0$ of the four-component proof of Theorem 5.5, where the truncation is read *at the level of the chiral spectrum* and not merely at the level of the field strength: the three-component theory is the GG-6 theory in which no fermion species carries a non-trivial $U(1)_{Q_i}$ charge. This distinction matters, since a spectrum with $F_{Q_i} = 0$ but with fermions still formally charged under $U(1)_{Q_i}$ would in principle generate mixed gravitational – $U(1)_{Q_i}$ and Abelian – $U(1)_{Q_i}$ anomaly contributions; the admissibility argument is that the three-component bulk has no such residual charges to begin with, so all $U(1)_{Q_i}$ -coupled anomaly slots are identically zero. Under that reading, the BI-6 four-form $X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - (2/5) F_B^2$ remains a closed, gauge-invariant four-form; the microscopic anomaly polynomial still factorizes as $I_8 = \frac{1}{2} X_4^2$; descent against a single two-form B_2 cancels the local consistent anomaly; and global consistency holds because the homotopy facts $\pi_6(SU(5)) = 0$ (because $6 \leq 2n - 1 = 9$ for group size $n = 5$ places π_6 in the stable range, where Bott periodicity gives $\pi_{2j}(SU) = 0$; note that the unstable $\pi_{2n}(SU(n)) = \mathbb{Z}_{n!}$ does *not* vanish, so the stable-range bound is what is doing the work here) and $\pi_6(U(1)) = 0$ (Abelian factors contribute nothing to π_6) together give $\pi_6(SU(5) \times U(1)_B) = 0$, so that no Witten-type six-dimensional gauge obstruction obstructs the three-component bulk. Anomaly freedom of the four-component bulk is the content of Theorems 5.5 and 6.2 of the present paper as written. Both bulks therefore satisfy POS-3 (anomaly-free admissibility by inflow); both satisfy AX-2 (GGEP) under the trace-ratio assignment of $-2/5$ to every Abelian slot; both satisfy POS-6 (Occam Gauge Normalization) under the OGN ledger at $L = 5$, since L is fixed by the size of the $SU(5)$ fundamental representation and is independent of the number of Abelian factors. The remaining postulates — AX-1, AX-3, AX-4, POS-1, POS-2, POS-4, POS-5, POS-7 — are satisfied trivially by both bulks because they constrain the geometry, the bulk-to-boundary projection, the symmetry-minimality content, and the falsifiability structure rather than the number of Abelian factors. Hence both bulks are constitutionally admissible. ■

Worked example: an explicit three-component reservoir. Constitutional admissibility is a statement about the axioms, and the proof above settles it. It is worth exhibiting an anomaly-free three-component spectrum as well, because the shortest route to one does not work and the reason it fails is instructive.

That route is truncation: take the benchmark $\mathcal{M}_{\text{gauge}}(25)$ of Section 7 and strike every species carrying non-trivial $U(1)_{Q_i}$ charge. Both members of every diagonal orbit go, and one member of every axial orbit, leaving $8\mathcal{C}(1/3, 0) \oplus 4\mathcal{C}(2/3, 0)$. The mixed monomials $\text{tr } R^2 F_B F_{Q_i}$ and $F_B^3 F_{Q_i}$ do vanish, as the truncation intends. The quartic Abelian balance does not survive. Matching (216) against $\frac{\Omega'}{2}(X_4^{(3)})^2$, the $p_1 \text{Tr } F_5^2$ and $(\text{Tr } F_5^2)^2$ monomials of the residue force $\Omega' = 3$; the $p_1 F_B^2$ and $\text{Tr } F_5^2 F_B^2$ monomials then force the Abelian coefficient $\gamma = 4/3$; and those two values demand an F_B^4 coefficient of $8/3$, against the $4/3$ the residue actually carries. No rescaling of γ repairs both, because removing a core block removes its x^4 and x^2 contributions in a fixed ratio. Truncation preserves the vanishing conditions and breaks the matching conditions.

The construction has to be carried out directly, and it closes. Write a three-component reservoir as a multiset of core blocks $\mathcal{C}(q_B, 0)$ with $q_B = n/3$ and signed multiplicities c_n , and set $N = \sum_n c_n$, $S_2 = \sum_n c_n q_B^2$, $S_4 = \sum_n c_n q_B^4$. Matching (216) against $\frac{\Omega'}{2}(X_4^{(3)})^2$ term by term gives $\Omega' = N/4$ from the $p_1 \text{Tr } F_5^2$ monomial and again, consistently, from $(\text{Tr } F_5^2)^2$; it gives $\gamma = 6S_2/N$ from $p_1 F_B^2$ and again from $\text{Tr } F_5^2 F_B^2$; and it leaves exactly one residual condition, from F_B^4 ,

$$N S_4 = 3 S_2^2. \quad (74)$$

Imposing in addition the BI-6 Abelian value $\gamma = 2/5$ — which Theorem 8.1 fixes from the $SU(5)$ trace ratio, and which is blind to the number of Abelian factors — the two conditions read $\sum_n c_n n^2 = \frac{3}{5}N$ and $\sum_n c_n n^4 = \frac{27}{25}N$. The smallest solution on the charge set $\{0, 1/3, 2/3\}$ is

$$\mathcal{M}_{\text{gauge}}^{(3)}(25) = 52\mathcal{C}(0, 0) \oplus 44\mathcal{C}(1/3, 0) \oplus 4\mathcal{C}(2/3, 0), \quad (75)$$

with $N = 100$ and $\Omega' = 25$. Its gauge and mixed monomials reproduce $\frac{25}{2}(X_4^{(3)})^2$ exactly, and the gravitational sector closes under the same law as the four-component case, $n_{\text{SD}} = 4\Omega' = 100$ and $n_F = 32\Omega' = 800$ (Appendix H). Baryon-neutral core blocks are not a new ingredient: the four-component benchmark already contains twelve of them, as the $\mathcal{C}(0, n/3)$ members of its axial orbits.

Two features of (75) deserve recording. Integrality of Ω' forces $\Omega' \in 25\mathbb{Z}$ here as well, by the same $L^2 = 25$ that governs the four-component case (Theorem 8.3); the minimal normalization is therefore a property of the $SU(5)$ block and not of the number of Abelian factors. And the minimal three-component reservoir is chirality-uniform, whereas the minimal four-component one is not: the benchmark $\mathcal{M}_{\text{gauge}}(25)$ requires $d_2 = -12$, an orbit taken with reversed chirality. Anomaly freedom does not prefer the four-component bulk. The preference argued below is empirical, and this subsection supplies no structural substitute for it.

Why $U(1)_B$ is essential to either bulk. The baryonic factor is structural for three independent reasons, none of which depend on the presence of $U(1)_{Q_i}$. First, it is the only exact gauge symmetry in the GG-6 particle ledger that can protect proton stability in a constitutionally sharp way. Second, the key trace ratio that later becomes the OGN-5 theorem compares the primitive baryonic direction to the traceless hypercharge-like splitting inside the unified five-dimensional block. In that sense the coefficient $2/5$ is not an arbitrary Abelian weight; it is the baryonic trace witness of the $3 + 2$

split inside the $SU(5)$ block. Third, the split-multiplet boundary construction requires a separate baryonic label precisely because unsplit $SU(5)$ multiplets cannot carry definite baryon number. Both the three-component and the four-component bulks therefore retain $U(1)_B$ as an essential factor.

Why $U(1)_{Q_i}$ is constitutionally optional. The additional informational Abelian factor is *allowed* but not *forced*. It is allowed because it is fully compatible with both local and global anomaly cancellation: Abelian factors have $\pi_6(U(1)) = 0$, so there is no independent Witten-type six-dimensional gauge obstruction; the BI-6 identity sees Abelian factors only through F^2 , so adding a second Abelian slot with the same coefficient preserves the factorized square structure rather than spoiling it; and in the global theorem it behaves as an anomaly-compatible spectator extension of the Abelian sector. It is not forced because, for the narrow purpose of proving complete anomaly freedom of BI-6, one could truncate to the three-component bulk and set $F_{Q_i} = 0$; the global cancellation theorem would still close. In that precise sense $U(1)_{Q_i}$ is constitutionally optional.

Why the four-component bulk is empirically preferred. After all, the test of an honest framework is what nature selects out of the constitutional admissible class, not what the constitution permits in the abstract. The four-component version is preferred over the three-component on three empirical and structural grounds.

- (1) *The Arisaka current equalities and the bioenergetic conversion factor Q_B .* The Arisaka equation $J_E/Q_B = J_e = J_B = J_{Q_i}$ developed in the companion papers (Arisaka, 2026, A2-GG6; Arisaka, 2026, A9-SM) requires $U(1)_{Q_i}$ to exist on the boundary as a fourth conserved current. Without $U(1)_{Q_i}$, the four current equalities collapse to three, and the bioenergetic conversion factor $Q_B \approx 0.1555$ eV per qubit — anchored by ATP hydrolysis at biological temperatures, one phosphoanhydride bond carrying two quanta, $2 N_A Q_B = 30.0$ kJ/mol against the measured 30.5 ± 0.5 kJ/mol (Arisaka, 2026, A7-Qi4) — has no constitutional home. The matching of Q_B to ATP hydrolysis is the sharpest bioenergetic anchor of the GG-Theory program; preserving it selects the four-component bulk.
- (2) *The OGN-5 dual-lock identity $K = LJ = M^2 + N^2 + L^2 = 35$.* Step 5 of the OGN-5 cascade in the companion paper (Arisaka, 2026, A2-GG6) establishes the dual-lock identity $K = LJ = M^2 + N^2 + L^2$ that fixes the master ledger integer K jointly with the closure index J . Under the equal-Abelian-treatment convention — which assigns the coefficient $-2/5$ symmetrically to every Abelian slot in the BI-6 four-form — the dual-lock identity closes arithmetically only for the four-component case $(M, N, L; J, K) = (1, 3, 5; 7, 35)$. The three-component bulk does not produce an integer dual-lock with the same small-integer character. The four-component bulk is therefore the choice that closes the OGN-5 ledger as a Diophantine cascade rather than as a free-parameter fit.
- (3) *The boundary Qi-4 pillars and the Equivariant Information Channel.* The boundary phenomenology of GG-Theory — the structural framing of life and mind as boundary projections of the $U(1)_{Q_i}$ current, articulated as the four pillars of GG-A7-Qi4 (Arisaka, 2026, A7-Qi4) — depends on the four-component version. Without $U(1)_{Q_i}$, the Equivariant Information Channel

(EIC) framework that the program uses to connect 6D bulk physics to biological information processing has no host symmetry. The four-component bulk is therefore the choice that allows the Qi-4 pillars to be derived rather than postulated.

In summary, both the three-component bulk and the four-component bulk are constitutionally admissible. The four-component bulk is preferred on empirical grounds: it carries the Arisaka current equalities and the bioenergetic conversion factor Q_B , it closes the OGN-5 dual-lock identity arithmetically, and it hosts the boundary Qi-4 pillars and the Equivariant Information Channel framework. The remainder of the present paper therefore works in the four-component bulk, with the understanding that every theorem and every proof restricts to the three-component bulk by the obvious truncation $F_{Q_i} = 0$. A reader who wishes to retain only the axiomatic content of the framework, deferring the empirical preference to the boundary papers, may read every BI-6 statement as a statement about the three-component bulk; the anomaly theorem and the OGN-5 ledger then close around $L = 5$ at $J = 7$ and the four-component dual-lock identity is replaced by its three-component cousin. A four-dimensional relative of the one-Abelian reading now exists in the literature: the anomaly-free baryonic completion of (Benakli, Goodsell, and Goudeau, 2026) gauges a single baryonic factor over the Standard-Model group (the $SU(5)$ organization forgone), with the anomaly canceled by added matter rather than by inflow, and with none of the empirical anchors enumerated above in play; Subsection 11.6 makes the comparison in full.

Cross-references. The companion paper Arisaka-GG-A2-GG6 (Arisaka, 2026, A2-GG6) cross-cites the present subsection at three points: at the opening statement that the constitution admits both three- and four-component bulks (Theorem 6 “Six forces from the constitution”), at the empirical adoption of the four-component bulk in the BI-6 coefficient theorem, and at the response to Objection 4 in the Common Objections section. Both $U(1)_{Q_i}$ -dependent companion papers — GG-A6-GRIP and GG-A7-Qi4 — inherit the four-component preference from the present subsection together with the empirical anchor.

6.7 Applicability of the Hopkins–Singer / Monnier–Moore framework to non-supersymmetric six-dimensional theories

A reader trained on the modern global-anomaly literature will reasonably ask whether the Hopkins–Singer / Monnier–Moore differential-cohomology framework, which we invoke in Section 6 and Appendix I to establish global anomaly freedom of BI-6, applies to a non-supersymmetric six-dimensional bulk such as GG-6. The question is non-trivial because Monnier and Moore (Monnier and Moore, 2019) developed the framework specifically for $\mathcal{N} = (1, 0)$ six-dimensional supergravity, and the supersymmetric structure plays a visible role in their explicit numerical evaluation of partition functions on bounding seven-manifolds.

The structural answer is that the framework’s four hypotheses (H1)–(H4), stated in full in Appendix I — (H1) the admissible mapping tori bound spin eight-manifolds over which the gauge and tensor data extend; (H2) no independent Witten-type gauge obstruction survives, π_6 vanishing for each factor of $SU(5) \times U(1)_B \times U(1)_{Q_i}$; (H3) the lattice-valued lift of the BI-6 four-form

obeys the Wu-shifted integrality law, so the exponentiated Wu–Chern–Simons functional is single-valued; and (H4) the four-form X_4 lifts to a globally defined degree-four differential cocycle — are purely bordism-theoretic / differential-cohomological statements about the bundle structure of the underlying manifold, independent of whether the matter content on that bundle is supersymmetric. The foundational construction of Hopkins and Singer ([Hopkins and Singer, 2005](#)) pre-dates the Monnier–Moore SUGRA specialization and was developed for general principal bundles over Spin manifolds; the four hypotheses inherit that generality. What *is* specific to $\mathcal{N} = (1, 0)$ in Monnier–Moore (2019) is the explicit evaluation of the eta-invariant in terms of the SUGRA spectrum on closed Spin seven-manifolds; that calculation is recorded here as Open Problem O9 (Section 17) for the non-supersymmetric cohort spectrum and is the natural follow-up computational program on top of the structural theorem. The structural verification of H1–H4 carried out in Section 6 does not invoke supersymmetry; the global theorem is therefore established at the structural level on the cohort GG-6 background class, with the explicit numerical confirmation of partition-function single-valuedness deferred to follow-up work that adapts the Monnier–Moore eta-invariant calculations to the non-supersymmetric setting.

7 Model-level benchmark spectrum

The local and global theorems of Sections 5–6 prove anomaly cancellation *assuming* the total polynomial factorizes as $I_8 = \frac{\Omega}{2} X_4^2$. The natural question is whether any finite chiral spectrum actually achieves this factorization. This section constructs one explicit solution—a *benchmark spectrum*—thereby proving that the structural theorem is satisfiable. The construction proceeds in four stages: three no-go results that constrain the search space, a modular “core block” that serves as the building unit, an orbit architecture that cancels all forbidden terms by design, and a gravitational completion by gauge-singlet tensors and fermions. The full computations appear in Appendices F–H.

7.1 Three no-go results

Before constructing a solution, three constraints narrow the search.

Lemma 7.1 (Species-wise diagonal no-go). *If $q_{Qi,s} = q_{B,s}$ for every species s , then all off-diagonal Abelian anomaly terms force all diagonal Abelian coefficients to vanish, contradicting BI-6.*

Lemma 7.2 (Fermion-only gravitational no-go). *Ordinary Weyl fermions alone cannot produce a factorized gravitational anomaly with vanishing $\text{tr } R^4$ and nonzero $(\text{tr } R^2)^2$.*

Lemma 7.3 (Split-multiplet necessity). *Unsplit bulk SU(5) multiplets cannot carry definite baryon number, because all components share a single $U(1)_B$ charge. Visible Standard-Model zero modes require split carriers with orbifold boundary conditions.*

These three results rule out the simplest constructions and motivate the orbit-based approach below.

7.2 Core block construction

The idea is to find a minimal $SU(5)$ multiplet package whose irreducible and cubic anomaly indices vanish *internally*, so that the only surviving contributions are the reducible terms needed for the BI-6 factorization. The following combination achieves this.

Definition 7.4 (Core block).

$$\mathcal{C}^\sigma(q_B, q_{Qi}) := \sigma \left[\mathbf{10} \oplus \mathbf{5} \oplus 2\bar{\mathbf{5}} \oplus 11 \cdot \mathbf{1} \right]_{(q_B, q_{Qi})}. \quad (76)$$

Here $\sigma = \pm 1$ is the six-dimensional chirality, and all components carry the same Abelian charges (q_B, q_{Qi}) . The 11 singlets are included to make the total dimension a perfect square: $d_C = 10 + 5 + 10 + 11 = 36 = 6^2$.

Proposition 7.5 (Core block anomaly data). *The core block has dimension $d_C = (L+1)^2 = 36$, quadratic index $A_C = L+1 = 6$, reducible quartic coefficient $C_C = 3$, and irreducible quartic and cubic indices*

$$B_C(L) = A_{3,C}(L) = L - 5, \quad (77)$$

both of which vanish at $L = 5$. The vanishing of B_C and $A_{3,C}$ is the key property: it ensures that the core block contributes only to the reducible anomaly polynomial, which is exactly the part that the Green–Schwarz mechanism can cancel. The proof is a direct computation given in Appendix F.

7.3 Orbit families and the Diophantine system

The six forbidden monomials (V1)–(V6) listed in Appendix D contain odd powers of the Abelian field strengths. These must cancel in the total spectrum. The orbit construction achieves this by pairing core blocks with opposite or exchanged Abelian charges, so that the odd-power terms cancel within each pair. Two pairing patterns are possible:

Definition 7.6 (Axial orbit). $A_n^\sigma := \mathcal{C}^\sigma(n/3, 0) \oplus \mathcal{C}^\sigma(0, n/3)$. One block carries baryon charge only, the other carries informational charge only. Every mixed monomial $F_B^a F_{Qi}^b$ with $a, b > 0$ vanishes on this pair because the two blocks have disjoint Abelian support.

Definition 7.7 (Diagonal orbit). $D_n^\sigma := \mathcal{C}^\sigma(n/3, n/3) \oplus \mathcal{C}^\sigma(n/3, -n/3)$. Both blocks carry the same baryon charge, but opposite informational charges. Mixed monomials odd in q_{Qi} vanish by the sign flip $q_{Qi} \rightarrow -q_{Qi}$.

The gauge-matter reservoir $\mathcal{M}_{\text{gauge}}(\Omega) = \bigoplus_n a_n A_n^+ \oplus \bigoplus_n d_n D_n^+$ satisfies the BI-6 factorization condition if and only if the following four equations hold (derived in Appendix G):

$$\sum_n a_n + \sum_n d_n = 2\Omega, \quad (78)$$

$$\sum_n n^2 a_n + 2 \sum_n n^2 d_n = 12\Omega/5, \quad (79)$$

$$\sum_n n^4 a_n = 72\Omega/25, \quad (80)$$

$$\sum_n n^4 d_n = 18\Omega/25. \quad (81)$$

The first equation (78) counts the total number of core blocks (the $\text{Tr } F_5^2$ coefficient). The second (79) matches the mixed gauge–Abelian coefficient $\text{Tr } F_5^2 \cdot F_B^2$. The third and fourth, (80)–(81), match the pure Abelian quartic coefficients $(F_B^2)^2$ and $(F_{Q_i}^2)^2$ separately; they differ because the axial and diagonal orbits distribute their Abelian charges differently between the B and Q_i directions.

Theorem 7.8 (T-BI6-8.8: Exact orbit solution). *For every $\Omega \in L^2\mathbb{Z} = 25\mathbb{Z}$, the system has the exact integer solution*

$$a_1 = \frac{8\Omega}{25}, \quad a_2 = \frac{4\Omega}{25}, \quad d_1 = \frac{48\Omega}{25}, \quad d_2 = -\frac{12\Omega}{25}, \quad d_3 = \frac{2\Omega}{25}. \quad (82)$$

The negative multiplicity $d_2 < 0$ is not a sign error: it means the $n = 2$ diagonal orbit enters with the opposite six-dimensional chirality, $\sigma = -1$ instead of $+1$. Equivalently, $|d_2|$ copies of D_2^- replace $|d_2|$ copies of D_2^+ . In six-dimensional chiral theories, a Weyl fermion of chirality $+1$ in representation ρ is physically distinct from a Weyl fermion of chirality -1 in the same representation — they contribute to the anomaly polynomial with opposite signs. Chirality-flipped species are standard in the 6D model-building literature (Green, Schwarz, and West, 1985, Sagnotti, 1992) and appear routinely in the Kumar–Morrison–Taylor enumeration of anomaly-free spectra (Kumar, Morrison, and Taylor, 2010). This chirality flip is the mechanism by which the Diophantine system achieves the non-trivial balance between the quartic Abelian coefficients.

7.4 Gravitational completion

The orbit construction of the previous subsection fixes the gauge-matter sector but leaves the purely gravitational anomaly unfixed. The anomaly polynomial of Weyl fermions has a universal gravitational piece $\propto (7p_1^2 - 4p_2)$, and the target $\frac{\Omega}{2}X_4^2$ contains a $(\text{tr } R^2)^2 \propto p_1^2$ term but no independent p_2 term. As shown in Proposition 5.10, fermions alone cannot match both coefficients simultaneously. The gravitational balance requires gauge-singlet *anti-self-dual tensors* (which contribute $\propto p_1^2 - 7p_2$ with a different p_2/p_1^2 ratio) and additional singlet fermions.

The resulting 2×2 linear system (derived in Appendix H) determines the tensor and fermion counts uniquely and linearly in Ω :

$$n_{\text{SD}} = 4\Omega, \quad n_F = 32\Omega; \quad \Omega = 25 : (100, 800); \quad \Omega = 50 : (200, 1600), \quad (83)$$

where the n_F gauge-singlet Weyl fermions carry the six-dimensional chirality opposite to the gauge reservoir (the same chirality-flip bookkeeping as the D_2^- orbit of Theorem 7.8), and exactly eight singlet fermions accompany each tensor: the structural ratio $n_F/n_{\text{SD}} = 8$ holds identically in Ω . The global phase of the chiral tensor sector is controlled by $n_{\text{SD}} \bmod 8$: the exponentiated eta invariant of n_{SD} chiral two-forms on a closed seven-manifold is a signature-type phase that trivializes manifestly when $n_{\text{SD}} \equiv 0 \pmod{8}$ (Alvarez-Gaumé and Witten, 1984, Monnier and Moore, 2019). Since $n_{\text{SD}} = 4\Omega$, this shortcut applies exactly when Ω is even: the $\Omega = 50$ branch ($n_{\text{SD}} = 200$) is manifestly signature-friendly, while on the AX-4-minimal branch $\Omega = 25$ ($n_{\text{SD}} = 100$) the tensor-sector phase vanishes identically on the intended-class fillings, where $\sigma(X_6 \times D^2) = 0$ (Lemma I.10 of Appendix I). Between the two branches, AX-4 (MSMF) favors the smaller reservoir: $\Omega = 25$ with

$n_{SD} = 100$ and $n_F = 800$. Whether this minimal branch is the unique physical realization remains an open question (the Minimal BI-6 Spectrum Problem, entry O8 of the triage in Section 17).

7.5 Non-supersymmetric gravitational anomaly closure: what plays the role of the gravitino

The standard six-dimensional $\mathcal{N} = (1, 0)$ supergravity anomaly cancellation (Sadov, 1996, Taylor, 2011) rests on the celebrated linear constraint

$$n_H - n_V + 29 n_T = 273, \quad (84)$$

which closes the irreducible $\text{tr } R^4$ part of the gravitational anomaly polynomial through the combined contributions of the graviton, the gravitino, the dilatino, the self-dual two-form of the gravity multiplet, and the matter hypermultiplets (n_H) and vector multiplets (n_V); the constant 273 is the gravitino-plus-gravity-multiplet contribution to the $\text{tr } R^4$ coefficient, derived in Alvarez-Gaumé and Witten (1984) and tabulated cleanly in Schwarz (1996) and Erler (1994).

The cohort framework lacks the gravitino, dilatino, and SUGRA gravity multiplet, so the constraint (84) does not apply verbatim to BI-6. A reader trained on 6D supergravity will reasonably ask: what plays the role of the gravitino in canceling the irreducible $\text{tr } R^4$ piece for the cohort spectrum?

The answer is the linear combination of anti-self-dual tensors and gauge-singlet Weyl fermions that the cohort framework supplies via AX-4 (MSMF) on the bulk gauge content. The per-field contributions to the gravitational anomaly polynomial are, from Alvarez-Gaumé and Witten (1984):

- Per six-dimensional Weyl fermion of chirality $\sigma \in \{\pm 1\}$: $\sigma \frac{1}{5760} (7 p_1^2 - 4 p_2) + (\text{Casimir terms})$.
- Per six-dimensional anti-self-dual two-form tensor: $-\frac{1}{360} (p_1^2 - 7 p_2)$, with the overall sign opposite to the fermion-aligned chirality of (262) (Alvarez-Gaumé and Witten, 1984).
- No gravitino, no dilatino contributions: 0, 0.

The target anomaly polynomial $(\Omega/2) X_4^2$ is, by Lemma 4.1, a polynomial in $(\text{tr } R^2)^2 \propto p_1^2$ alone — it has no irreducible p_2 piece. The closure of the p_2 coefficient is therefore a single linear equation in $(n_F^{\text{eff}}, n_{SD})$ where n_F^{eff} is the chirality-signed total Weyl-fermion count of the spectrum and n_{SD} is the number of gauge-singlet anti-self-dual tensors. Combined with the matching condition for the p_1^2 coefficient, this becomes the 2×2 linear system written out explicitly in Appendix H.

The system has a unique solution (given the visible gauge spectrum fixed in the core-block construction (Section 7.2) and the orbit-family Diophantine system (Section 7.3)): for the minimal benchmark $\Omega = 25$, the gravitational completion requires $n_{SD} = 100$ anti-self-dual tensors and $n_F = 800$ gauge-singlet Weyl fermions of the opposite chirality; for the next branch $\Omega = 50$, $n_{SD} = 200$ and $n_F = 1600$. On the $\Omega = 50$ branch n_{SD} is divisible by 8, so the Atiyah–Patodi–Singer signature phase of the tensor sector trivializes manifestly; on the $\Omega = 25$ branch the phase is settled by the Wu–Chern–Simons quadratic refinement (see Monnier and Moore, 2019; Appendix I); AX-4 then selects the $\Omega = 25$ minimal branch as the cohort canonical realization, with the larger reservoirs being unforced by the constitution.

The role of the gravitino is therefore played in the cohort framework by the precise linear balance between anti-self-dual tensors and singlet Weyl fermions enforced by AX-4 on the visible spectrum. No supersymmetry is required. The cohort spectrum is not a non-supersymmetric *evasion* of the SUGRA 273-constraint; it is a different spectrum that satisfies a different but equally tight linear constraint, with the constitutional clauses doing the selection that supersymmetry would have done.

7.6 Model-level theorem

Theorem 7.9 (T-BI6-8.9: Model-level BI-6 anomaly verification). *(For the complete set of admissible reservoirs, of which the spectrum below is one member, see Remark 7.10.) For every $\Omega \in 25\mathbb{Z}$, the explicit spectrum $\mathcal{M}_{\text{full}}(\Omega) = \mathcal{M}_{\text{gauge}}(\Omega) \oplus \mathcal{G}_{\text{grav}}(\Omega) \oplus \mathcal{M}_{\text{vis}}^{\text{vec}}$ satisfies*

$$\boxed{I_8^{\text{tot}} = \frac{\Omega}{2} X_4^2}, \quad (85)$$

with all irreducible and forbidden terms vanishing exactly.

Remark 7.10 (Existence, and the complete solution set). Theorem 7.9 exhibits one explicit finite spectrum that realizes the BI-6 factorization, and the benchmark is not unique, minimal, or physically preferred. More can now be said, and it upgrades the result from a witness to a classification. The four matching equations (78)–(81) are a linear system in five multiplicities, so their solution set is an affine line, and that line can be written down in closed form for an arbitrary Abelian coefficient γ :

$$\begin{aligned} a_1 &= 6\Omega\gamma^2 - 8\Omega\gamma + \frac{64}{15}\Omega - \frac{64}{3}d_3, & a_2 &= \frac{3}{4}\Omega\gamma^2 + \frac{1}{2}\Omega\gamma - \frac{4}{15}\Omega + \frac{4}{3}d_3, \\ d_1 &= -\frac{15}{2}\Omega\gamma^2 + 8\Omega\gamma - \frac{32}{15}\Omega + \frac{77}{3}d_3, & d_2 &= \frac{3}{4}\Omega\gamma^2 - \frac{1}{2}\Omega\gamma + \frac{2}{15}\Omega - \frac{20}{3}d_3, \end{aligned} \quad (86)$$

with d_3 free. At $\gamma = 2/5$ this reduces to the family (238)–(241) printed in Appendix G, and at $d_3 = 2\Omega/25$ to the benchmark (82). So the admissible gauge reservoirs are *exactly* the integer points of (86), and the benchmark is the self-similar branch of a known family rather than a fortunate find.

Two consequences are worth recording, since both cut against a natural over-reading. The first is that the count equation (78) contains no γ ; the signed reservoir dimension is therefore the same for every Abelian coefficient, and with it the gravitational deficit and the completion $(n_{\text{SD}}, n_F) = (4\Omega, 32\Omega)$. The gravitational books, as this Part has said, see only the count and the chirality of what propagates — which is precisely why they cannot see γ . The second is that (86) is solvable in integers for every rational γ , so the existence of an anomaly-free spectrum does not distinguish the BI-6 value. It is fixed by the trace-ratio identity of Theorem 8.1, and by nothing in this Part. The benchmark is a transparent, verifiable construction whose purpose is to demonstrate that the structural factorization condition of Theorem 5.5 is not vacuous—it is satisfiable by a concrete chiral reservoir. Whether a smaller or more economical realization exists is a separate mathematical question.

Remark 7.11 (Normalization and the link to Theorem 5.5). The model-level theorem produces $I_8 = (\Omega/2) X_4^2$ rather than the primitive $I_8 = \frac{1}{2}X_4^2$. These are equivalent at the level of anomaly cancellation: the overall factor Ω is absorbed into the normalization of the tensor kinetic term and the Green–Schwarz coupling κ . The structurally important fact is that I_8 factorizes as a

perfect square through X_4^{BI6} with the coefficient vector $(1, -1, -2/5, -2/5)$, with all irreducible and forbidden terms vanishing exactly. This discharges proof obligations **(PO1)**–**(PO4)** of Section 5.4; **(PO5)**–**(PO6)** are discharged by Theorem 5.5 and its Corollary; **(PO7)**–**(PO8)** are discharged by Theorem 6.2 in Section 6. Note that the global theorem (including the Wu-shift integrality condition of hypothesis H3) uses the primitive integral class $\hat{X} = \frac{5}{8\pi^2} X_4 \in H^4(M_6, \mathbb{Z})$, which is defined independently of Ω . The value of Ω affects the size of the benchmark reservoir but not the integrality or quadratic-refinement properties of the BI-6 cocycle. The global anomaly theorem is therefore valid for every $\Omega_{\text{orb}} \in 25\mathbb{Z}$, not only for the minimal branch $\Omega_{\text{orb}} = 25$. (The *effective* normalization $\Omega_{\text{eff}} = \Omega_{\text{orb}}/L^2$, which is what the late-time descent of Section 7.7 drives to unity, is a different quantity; the bare symbol Ω always means Ω_{orb} in this appendix.)

7.7 The Green–Schwarz normalization Ω : static multiplicity versus late effective normalization

Two logically different uses of the symbol Ω must be separated if the BI-6 theorem and the later GDOS applications are to remain conceptually clean.

Static reservoir multiplicity. The Diophantine benchmark of Section 7 proves the existence of an anomaly-matching reservoir with integer orbit multiplicities whenever

$$\Omega_{\text{orb}} \in 25\mathbb{Z}.$$

This Ω_{orb} is a *static multiplicity label* for the microscopic benchmark reservoir. It tells us how many copies of the primitive orbit packet appear in a given arithmetic solution.

Primitive anomaly normalization. The anomaly theorem itself is written in primitive normalization,

$$I_8 = \frac{1}{2} X_4^2,$$

with the global counterterm built from the primitive integral lift $[\hat{X}]$. Therefore the complete local and global anomaly theorem does *not* require a separate proof that $\Omega_{\text{orb}} = 1$. The existence theorem for the benchmark reservoir and the primitive theorem of anomaly freedom are logically distinct statements.

Late operational normalization. The physically observed low-energy normalization is not the microscopic reservoir count itself. In GDOS one observes a coarse-grained boundary packet produced by OGGEF. The relevant symbol is therefore a different quantity, denoted here by Ω_{eff} .

Condensed GDOS attractor sketch. Let the primitive GG-6 benchmark packet decompose into L^2 symmetry-equivalent microscopic cells, with $L = 5$ from the OGN-5 theorem. Assume the OGGEF boundary channel registers the normalization carried by one operationally coarse-grained cell, while the full microscopic reservoir remains conserved in the bulk. Then the late-time GDOS

attractor implies

$$\Omega_{\text{eff}}(\mu) = \Omega_{\text{orb}} w_*(\mu),$$

where $w_*(\mu)$ is the boundary weight of one representative cell. Under DGI, the total primitive packet is conserved; under GSSB, the L^2 microscopic cells become boundary-distinguishable; under GA, symmetry-equivalent cells are driven to the equal-weight fixed point

$$w_*(\mu) \longrightarrow \frac{1}{L^2}.$$

Hence

$$\Omega_{\text{eff}} \longrightarrow \frac{\Omega_{\text{orb}}}{L^2}.$$

At the primitive benchmark $\Omega_{\text{orb}} = L^2 = 25$, one obtains

$$\Omega_{\text{eff}} \longrightarrow 1.$$

Physics intuition. The microscopic anomaly reservoir counts how many primitive orbit packets exist in the six-dimensional bulk. The boundary observer does not measure that raw count directly. OGGEF first projects the bulk packet to an operational channel; GDOS then subjects the channel to the standard open-system sequence $\text{DGI} \rightarrow \text{GSSB} \rightarrow \text{GA}$. The first step preserves the total packet, the second resolves symmetry-equivalent cells, and the third contracts their weights toward the unique symmetric attractor. Therefore the boundary normalization is the attractor quotient of the microscopic reservoir, not the raw reservoir multiplicity itself.

Why the distinction is drawn here. The global anomaly theorem of BI-6 does not depend on the late-time claim $\Omega_{\text{eff}} \rightarrow 1$. Nevertheless, the present sketch is included so that the paper is self-contained on the logical relation between the static anomaly normalization and the operational normalization used later in GDOS and cosmology. The anomaly theorem proves the primitive statement; GDOS explains why the observed effective normalization is the attractor image of that primitive statement.

What Part II has established, and on what. The title claim of this paper is now proved, and it is worth separating what was proved from what was assumed, because the two are easy to blur.

Locally, the cancellation is exact and unconditional. Given the four-component identity BI-6 and the matter content the constitution specifies, the one-loop anomaly polynomial factorizes as a perfect square, the irreducible and mixed terms vanish, and the single tensor field cancels what remains. Nothing in that argument is approximate and nothing in it is a choice; the coefficients that make it work are the ones the trace ratio hands over, and there is no free parameter anywhere in the calculation.

Globally, there are two results, at two strengths, and the difference matters. Theorem 6.2 proves that the four hypotheses of the differential-cohomology framework hold automatically on the intended GG-6 background class — the product and collar geometries the theory actually uses — so the global phase is trivial there. Theorem 6.9 then removes the topological fine print: the relevant

spin bordism groups vanish, so the same conclusion holds on every compact spin base and for large gauge transformations of any topological type. The second theorem is the stronger claim, and it closes a question this program had previously listed as open.

The benchmark spectrum does something narrower than either, and the narrowness is the point. It proves the theorem is not vacuous: an explicit finite chiral spectrum exists that saturates the factorization, with a definite number of fermions in definite representations and a gravitational sector that closes. It does *not* prove that this spectrum is the one nature chose, and it does not prove it is the smallest. Whether the minimal branch is the unique physical realization is recorded as an open problem, not quietly assumed.

One honest limitation belongs here rather than in a footnote. The anomaly test is passed by a whole family of Abelian coefficients, not by the BI-6 value alone; Remark 7.10 says so explicitly and the proof is written to make the dependence visible. What singles out the BI-6 value is not consistency but the constitution — specifically the trace ratio of the unified block, which is the subject of Part III. A reader who wants to know why the coefficients are what they are should read on; a reader who wants to know whether the theory is consistent has the answer already.

Part III

OGN-5 Arithmetic

The integer ledger as a theorem, not an input

The five integers $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ were introduced in Arisaka-GG-A2-GG6 (Arisaka, 2026, A2-GG6) as a compact summary of the normalization freedom in the BI-6 identity. This Part demonstrates that they are not free parameters: every one of them is forced by the GG-6 gauge architecture and the BI-6 trace-ratio structure. Section 8 shows how OGN-5 determines the anomaly-matching arithmetic, and Section 9 upgrades the entire packet to theorem level. The complete derivation is given in Appendix E.

The claim in this Part is unusual enough to be worth stating carefully before the machinery starts. Most theories of fundamental physics contain numbers that have to be measured and then put in by hand: the Standard Model has twenty-eight of them. The proposal here is that five small integers — one, three, five, seven, thirty-five — are not inputs of that kind. They are forced, each by an arithmetic condition that the previous one has already fixed, and the chain starts from the anomaly identity proved in Part II rather than from any measurement.

What makes such a claim checkable rather than mystical is that the conditions are ordinary Diophantine equations — equations required to have whole-number solutions — and whole-number conditions are unforgiving. A quantity that must be an integer either is one or is not; there is no adjusting it slightly to fit. So the reader’s task in this Part is not to be persuaded but to check that each step really is forced and that no measured quantity has been smuggled in as a premise. The text tries to make that easy: at each step it names what is being consumed, and where a step rests on a constitutional clause rather than on arithmetic alone, it says so in that sentence rather than in a caveat later.

Two of the five integers do most of the visible work downstream: $L = 5$, which is the size of the unified block, and $J = 7$, which is what makes the inverse fine-structure constant come out to $2^7 + 3^2 = 137$ at leading order. Both are derived here.

8 OGN-5 arithmetic and the spectrum problem

8.1 The BI-6 coefficient as a function of (N, L)

The BI-6 Abelian coefficient is not an independent parameter. It is a *consequence* of the OGN-5 integers.

Theorem 8.1 (T-BI6-9.1: OGN determination of the BI-6 coefficient). *For a gauge group $SU(L)$*

with the standard embedding of $SU(N) \times SU(L-N) \times U(1)_Y$, the Abelian trace ratio is

$$\frac{\text{Tr}(B_0^2)}{\text{Tr}(Y_0^2)} = \frac{L-N}{L}. \quad (87)$$

For $(N, L) = (3, 5)$: the BI-6 coefficient is $(5-3)/5 = 2/5$.

Proof. The hypercharge generator in the fundamental $\mathbf{L}$ is

$$Y_0 = \text{diag}\left(-\frac{1}{N}, \dots, -\frac{1}{N}, \frac{1}{L-N}, \dots, \frac{1}{L-N}\right)$$

with N entries of $-1/N$ and $L-N$ entries of $1/(L-N)$. The baryon generator is

$$B_0 = \text{diag}(1/N, \dots, 1/N, 0, \dots, 0).$$

Then

$$\text{Tr}(Y_0^2) = \frac{N}{N^2} + \frac{L-N}{(L-N)^2} = \frac{1}{N} + \frac{1}{L-N} = \frac{L}{N(L-N)}, \quad (88)$$

and

$$\text{Tr}(B_0^2) = \frac{N}{N^2} = \frac{1}{N}. \quad (89)$$

The ratio is $\text{Tr}(B_0^2)/\text{Tr}(Y_0^2) = N(L-N)/(NL) = (L-N)/L$. ■

8.2 Core block determined by L

Theorem 8.2 (T-BI6-9.2: OGN determination of the core block). *The core block (76) has the following data expressed entirely in terms of the OGN integer L :*

$$d_{\mathcal{C}} = (L+1)^2, \quad (90)$$

$$A_{\mathcal{C}} = L+1, \quad (91)$$

$$n_{\text{singlets}} = \frac{L^2 - L + 2}{2}, \quad (92)$$

$$\frac{d_{\mathcal{C}}}{A_{\mathcal{C}}} = L+1. \quad (93)$$

For $L = 5$: $d_{\mathcal{C}} = 36$, $A_{\mathcal{C}} = 6$, $n_{\text{singlets}} = 11$, $d_{\mathcal{C}}/A_{\mathcal{C}} = 6$.

Proof. The core block is $\mathcal{C} = \mathbf{L}(\mathbf{L}-\mathbf{1})/2 \oplus \mathbf{L} \oplus 2\bar{\mathbf{L}} \oplus n_1 \cdot \mathbf{1}$ where n_1 is chosen so that $d_{\mathcal{C}} = (L+1) \cdot A_{\mathcal{C}}$. In the convention used throughout this paper, where $A(\mathbf{5}) = 1$ (i.e., $A_{\rho} = 2T_2(\rho)$), the quadratic index is $A_{\mathcal{C}} = (L-2)+1+2 = L+1$. The dimension is $d_{\mathcal{C}} = L(L-1)/2 + 3L + n_1$. Setting $d_{\mathcal{C}} = (L+1)^2$ gives $n_1 = (L+1)^2 - L(L-1)/2 - 3L = (L^2 - L + 2)/2$. For $L = 5$: $n_1 = (25 - 5 + 2)/2 = 11$. ■

8.3 Minimum normalization determined by L

Theorem 8.3 (T-BI6-9.3: OGN determination of the minimum Ω). *The orbit solution (82) has denominators of L^2 . Therefore*

$$\Omega_{\min}^{\text{gauge/mixed}} = L^2 = 25. \quad (94)$$

Remark 8.4 (Proposed lattice normalization). A sharper bound,

$$\Omega_{\min}^{\text{lattice}} = 2N^2L^2 = 450, \quad (95)$$

has been proposed for the charge-lattice well-definedness of the exponentiated Green–Schwarz term (proof obligation (PO8) of Section 5.4). Its derivation is not carried out in this paper, and no downstream result uses it. On the intended class, (PO8) is discharged instead by the integral lift and the Wu-shifted quadratic refinement of Appendix I (cf. Remark 7.11), which hold for every $\Omega \in 25\mathbb{Z}$, so the benchmark $\Omega = 25$ is not in tension with (PO8). Whether the sharper lattice normalization is required in the unrestricted category is folded into Open Problem O10.

8.4 Summary: what OGN-5 determines

Table 5: Everything the integer ledger determines, in one table. The second column gives each quantity as an expression in the ledger integers and the third the number that results. The point is the second column: each entry is arithmetic in integers with no continuous input anywhere, so a reader can recompute every value in the third column by hand. If a single ledger integer were different, every row would move together — which is what makes the ledger falsifiable rather than merely economical.

Quantity	OGN expression	Value
BI-6 Abelian coefficient	$(L-N)/L$	2/5
Core block dimension	$(L+1)^2$	36
Core block index	$L+1$	6
Singlets per core block	$(L^2-L+2)/2$	11
Minimum Ω (gauge/mixed)	L^2	25
Minimum Ω (lattice; proposed, Remark 8.4)	$2N^2L^2$	450
Charge lattice unit	$1/N$	1/3
OGN-5 ledger integer (second entry)	N	3
Color triplicity index — what that entry counts	$N \equiv N_c$	3
Generation count — <i>not</i> a ledger integer (GG-P2-N3)	N_g	3
Quark-flavor count at high scale	$N_f = 2N_g$	6
Gauge group rank	$L-1$	4
Spacetime dimension	$L+1$	6

Table 5 collects what the ledger determines within this paper. The residual freedom not determined by OGN-5 is limited to the gravitational completion sector (n_{SD}, n_F) and the overall normalization Ω . The former is a 2×2 linear system; the latter is subject to minimization by AX-4/POS-6 (MSMF/OGN).

9 OGN-5 as a theorem derived from BI-6

9.1 Precise meanings of the five integers

Definition 9.1 (Meaning of $M, N, L; J, K$). In the intended GG-6 category:

- (i) M is the **primitive tensor/gravitational slot count**. It counts the minimal Green–Schwarz tensor channel carried by the single BI-6 tensor lattice.
- (ii) N is the **color triplicity index**, written N_c wherever the physical color count rather than the ledger slot is meant. It is the size of the color block carried inside the defining representation of the unified group. It is *not* the number of fermion generations: that quantity is written N_g , it is not a member of this ledger, and it is fixed by an argument of an entirely different kind (Arisaka, 2026, P2-N3). That $N_c = N_g = 3$ is a coincidence on the boundary, and this paper never uses it.
- (iii) L is the **defining unified block size**. In GG-6 it is the defining dimension of the minimal simple unified group SU(5).
- (iv) J is the **minimal symmetric closure cardinality**: if the largest allowed absolute Abelian orbit level is N , then OGN closure together with sign symmetry requires the charge packet $\{0, \pm 1, \pm 2, \dots, \pm N\}$, and J is the cardinality of that packet, so $J = 2N + 1$. (Appendix E states the same definition; the two must agree, because Step 4 of Theorem 9.3 evaluates this one.) The symmetric, centered form of the charge packet is a structural claim and not a convention: among admissible closure packets it is the minimal one symmetric about a neutral slot, and the selection is AX-4 (MSMF) acting through POS-6 (OGN). J is accordingly not an independent integer of the ledger but a structural function of N ; what carries content is the minimality clause, which is constitutional and is named as such.
- (v) K is the **total lock index**. It is simultaneously the rectangular product LJ and the quadratic norm of the normalization packet.

Three counts, three symbols. Three integers of value three occur in this program and must not be conflated. $N_c = 3$ is the color count — the OGN-5 integer N of Definition 9.1, and the N_c of the QCD beta function. $N_g = 3$ is the number of fermion generations, which is not an OGN-5 member and is established in (Arisaka, 2026, P2-N3). N_f retains its standard meaning throughout, the number of quark flavors entering a beta function, and is 6 at high scale — so that $11N_c - 2N_f = 33 - 12 = 21$ reads as it always has. Nothing in the OGN-5 cascade uses N_g , and no step below consumes it.

9.2 The unified block size from single-tensor consistency alone

The one integer still taken as given. Of the five integers, L has until now been the one taken in from outside. The cascade reads it off the statement that SU(5) is the smallest simple group

containing the observed low-energy gauge content — and that statement already knows how many colors there are. An integer obtained that way cannot honestly be used, later in the same cascade, to conclude how many colors there are. The present subsection removes the circle at its source. It shows that $L = 5$ follows from the Green–Schwarz construction itself: no observed gauge group, no block split, and no appearance of N at any step.

The physical idea, before any algebra. A single two-form can only mop up an anomaly that has the shape of a perfect square. That is the entire content of the rank-one criterion (Theorem 5.1): what the tensor can absorb is $\frac{1}{2}(v \cdot x)^2$, where x collects the four *quadratic* curvature invariants $\text{tr } R^2$, $\text{Tr } F_5^2$, F_B^2 , $F_{Q_i}^2$.

Squaring a quadratic form is a restrictive operation, and it is worth pausing on how restrictive. It can manufacture $\text{Tr } F_5^2 \wedge F_B^2$, because that is a product of two of the four invariants. It can never manufacture $\text{Tr } F_5^4$ — meaning the genuinely irreducible quartic trace, the one that is not a product of two quadratic traces — because a product of two second-order objects is by construction reducible. And it can never manufacture $\text{Tr } F_5^3 \wedge F_B$ either, for a simpler reason still: X_4 contains no cubic invariant anywhere in it, so there is nothing in the square that could supply the $\text{Tr } F_5^3$.

Now read that as a statement about the matter rather than about the tensor. A chiral spectrum in six dimensions produces, in general, all of these monomials. Two of them have no channel. The bulk is not free to generate an irreducible $\text{Tr } F_5^4$, or a mixed $\text{Tr } F_5^3 \wedge F_B$, and then hope that the Green–Schwarz term will take care of them: there is no term available that could. They must cancel *among the fermions themselves*, before the inflow mechanism is invoked at all.

This is the point at which a familiar-looking requirement becomes a genuinely different one. In the usual model-building idiom one says “choose a spectrum whose anomalies cancel.” Here the single-tensor architecture says something sharper: two specific group-theoretic indices of the charged matter — the irreducible quartic index B and the cubic index A_3 — must vanish identically, as a *precondition* for single-tensor inflow rather than as a consequence of it. The present paper has been using that precondition all along; Section 7.2 builds its core block precisely so that “irreducible and cubic anomaly indices vanish internally.”

What has not been done until now is to decline to evaluate those two indices at a chosen block size, and to carry L as a symbol instead. Do that, and each condition becomes an equation in L . They turn out to be the same equation, and it has one root.

Theorem 9.2 (T-BI6-9.0: Unified block size from single-tensor consistency). *Let the charged core package of the bulk be*

$$\mathcal{P}(p, f, g) = p \Lambda^2 \oplus f \mathbf{L} \oplus g \bar{\mathbf{L}}, \quad p, f, g \in \mathbb{Z}_{\geq 0}, \quad (96)$$

built from the defining representation $\mathbf{L}$ of $\text{SU}(L)$, its conjugate, and the two-index antisymmetric Λ^2 , with all components carrying common Abelian charges and the package completed by $\text{SU}(L)$ singlets. Single-tensor Green–Schwarz consistency requires that the irreducible quartic index and the cubic index vanish on the package,

$$B[\mathcal{P}] = 0, \quad A_3[\mathcal{P}] = 0. \quad (97)$$

These two conditions hold if and only if

$$g = 2p, \quad f = p(6 - L). \quad (98)$$

Imposing in addition $p = 1$ (a single antisymmetric, by AX-4 MSMF), $f \geq 1$ (the package contains a defining representation), and chirality of the surviving package (which excludes $L \leq 4$; see Step 4 of the proof of Theorem 9.2 for the three cases), the unique solution is

$$\boxed{L = 5, \quad \mathcal{P} = \Lambda^2 \oplus \mathbf{5} \oplus 2\bar{\mathbf{5}} = \mathbf{10} \oplus \mathbf{5} \oplus 2\bar{\mathbf{5}}.} \quad (99)$$

Equivalently, for the core block $\mathcal{C}$ of Eq. (76),

$$B_{\mathcal{C}}(L) = A_{3,\mathcal{C}}(L) = L - 5. \quad (100)$$

Neither the statement nor the proof refers to a color count, to the boundary gauge content, or to any split of the defining representation.

Proof. Step 1: which indices the tensor cannot absorb. This is Theorem 5.3, proved in Part II: single-tensor inflow through a four-form quadratic in the curvatures can reach neither the irreducible quartic $\text{Tr } F_5^4$ nor the mixed cubic $\text{Tr } F_5^3 \wedge F_B$, so both must cancel on the spectrum itself. With all components of $\mathcal{P}$ carrying common Abelian charges the charge factors out of the cubic condition, leaving $A_3[\mathcal{P}] = 0$, and the quartic condition leaves $B[\mathcal{P}] = 0$. This is (97).

Step 2: the indices, computed from the weights. Let $H = \text{diag}(h_1, \dots, h_L)$ be a traceless diagonal generator and write $P_k := \sum_i h_i^k$, so that $P_1 = 0$. The weights of Λ^2 are $h_i + h_j$ with $i < j$, whence

$$\sum_{i < j} (h_i + h_j)^2 = (L - 2) P_2, \quad (101)$$

$$\sum_{i < j} (h_i + h_j)^3 = (L - 1) P_3 + 3(P_2 P_1 - P_3) = (L - 4) P_3, \quad (102)$$

$$\sum_{i < j} (h_i + h_j)^4 = (L - 1) P_4 + 4(P_3 P_1 - P_4) + 3(P_2^2 - P_4) = (L - 8) P_4 + 3 P_2^2. \quad (103)$$

In the normalization $A(\mathbf{L}) = 1$ used throughout this paper, and writing $\text{Tr}_\rho H^4 = B_\rho \text{Tr}_{\mathbf{L}} H^4 + C_\rho (\text{Tr}_{\mathbf{L}} H^2)^2$, these read

$$A(\Lambda^2) = L - 2, \quad A_3(\Lambda^2) = L - 4, \quad B(\Lambda^2) = L - 8, \quad C(\Lambda^2) = 3, \quad (104)$$

while for the defining representation and its conjugate $A = 1$, $A_3 = \pm 1$, $B = 1$, $C = 0$, the cubic index changing sign under conjugation and the quartic index not.

Two of these four values are already in use in the present paper and serve as a check on the normalization: (101) is what Theorem 8.2 uses when it writes $A_{\mathcal{C}} = (L - 2) + 1 + 2 = L + 1$, and the $3P_2^2$ of (103) is what gives the reducible coefficient $C_{\mathcal{C}} = 3$ of Proposition 7.5.

Step 3: the two conditions become one equation. Summing the indices over (96),

$$A_3[\mathcal{P}] = p(L - 4) + f - g, \quad (105)$$

$$B[\mathcal{P}] = p(L - 8) + f + g. \quad (106)$$

Setting both to zero and taking the difference gives $4p - 2g = 0$, hence $g = 2p$; taking the sum gives $2p(L - 6) + 2f = 0$, hence $f = p(6 - L)$. This is (98). Substituting $p = 1$, $f = 1$, $g = 2$ back into (105)–(106) gives (100): $A_{3,C} = (L - 4) + 1 - 2 = L - 5$ and $B_C = (L - 8) + 1 + 2 = L - 5$, the same linear function twice.

Step 4: the selection. With $p = 1$ the solution family is $f = 6 - L$, $g = 2$, one package for each L . What excludes $L \leq 4$ is chirality of the surviving package, and the three cases are different and are worth separating, because the criterion sometimes quoted — complexity of Λ^2 — is not by itself sufficient. At $L = 2$, Λ^2 is the singlet and the package carries no antisymmetric at all. At $L = 4$, Λ^2 is the self-conjugate **6** and $f = g = 2$ leaves the remaining sector vector-like. At $L = 3$, $\Lambda^2(\mathbf{3}) = \bar{\mathbf{3}}$ is complex, so complexity does not exclude it; what excludes it is that the package $\bar{\mathbf{3}} \oplus \mathbf{3}\mathbf{3} \oplus 2\bar{\mathbf{3}}$ is $\mathbf{3}\mathbf{3} \oplus \mathbf{3}\bar{\mathbf{3}}$, which is vector-like and contributes no net anomaly. (This case also disposes of an apparent competitor on the minimality count of Theorem 9.5: the $L = 3$ package carries eighteen charged states against the $L = 5$ package's twenty-five, and would win on AX-4 alone were it chiral.) Non-negativity of f gives $L \leq 6$, and $f \geq 1$ gives $L \leq 5$. Hence $L = 5$, and the package is $\mathbf{10} \oplus \mathbf{5} \oplus 2\bar{\mathbf{5}}$, which is Eq. (76). ■

The one competitor, and what excludes it. The hypothesis $f \geq 1$ is doing real work and should not be smuggled past the reader. Dropping it admits exactly one further package, at $L = 6$: $\Lambda^2 \oplus 2\bar{\mathbf{6}}$, that is $\mathbf{15} \oplus 2\bar{\mathbf{6}}$, with $f = 0$. It is genuinely chiral and it satisfies both preconditions. What separates it from $L = 5$ is AX-4 (MSMF) in its ordinary sense: the $L = 5$ package carries $10 + 5 + 10 = 25$ charged states against the $L = 6$ package's $15 + 12 = 27$, so the minimal chiral package consistent with single-tensor inflow is the SU(5) one. This is the same minimality clause that selects $k = 1$ in Theorem 9.5, used the same way and carrying the same status: a constitutional selection among admissible solutions, not a consequence of the equations. We say so rather than let the boxed result of Theorem 9.2 read as though the algebra alone had closed.

What this does and does not deliver. It delivers L . The unified block size is now fixed by the same single-tensor architecture that the rest of Part II runs on, and the OGN-5 cascade no longer needs the smallest-simple-group clause — which is to say, no longer needs to know the low-energy gauge group in order to derive the low-energy gauge group. It does *not* deliver N . The Abelian entry of the coefficient vector is $(L - N)/L$, a function of N whatever L may be, and nothing in the argument above touches it. The trace-ratio route to the color index remains what Section 9 says it is, and the $3 + 2$ split that route presupposes is not supplied here. What has changed is the honesty of the bookkeeping: one of the two integers that used to arrive from the boundary now arrives from the bulk, and the one that still arrives from the boundary is no longer able to hide behind it.

9.3 The trace-ratio theorem that fixes the BI-6 Abelian fraction

The identity used in this step is Theorem 8.1 of Section 8, proved there and carried out in full detail in Appendix E. It is restated here in the form the cascade consumes, and not re-derived a third time; a reader who wants the computation should go to one of those two places.

Let a general $SU(L)$ block split into a color block of size N and a complementary block of size $L - N$. Define the diagonal directions

$$B_{N,L} = \text{diag} \left(\underbrace{\frac{1}{N}, \dots, \frac{1}{N}}_N, \underbrace{0, \dots, 0}_{L-N} \right),$$

$$Y_{N,L} = \text{diag} \left(\underbrace{-\frac{1}{N}, \dots, -\frac{1}{N}}_N, \underbrace{\frac{1}{L-N}, \dots, \frac{1}{L-N}}_{L-N} \right).$$

The first is the primitive baryonic block direction; the second is the traceless hypercharge-like splitting direction. A direct trace computation gives

$$\text{Tr}(B_{N,L}^2) = \frac{1}{N}, \quad \text{Tr}(Y_{N,L}^2) = \frac{1}{N} + \frac{1}{L-N} = \frac{L}{N(L-N)}.$$

Therefore

$$\frac{\text{Tr}(B_{N,L}^2)}{\text{Tr}(Y_{N,L}^2)} = \frac{L-N}{L}. \quad (107)$$

This formula is the structural bridge from BI-6 to OGN-5. The BI-6 coefficient requires the Abelian fraction to be exactly $2/5$; hence the trace-ratio closure condition is

$$\frac{L-N}{L} = \frac{2}{5}. \quad (108)$$

Why the trace-ratio is canonical, not a normalization choice. A careful reader may worry that the ratio $(L-N)/L = 2/5$ depends on the specific generator normalization used for $B_{N,L}$ and $Y_{N,L}$ above, so that a different but equally natural normalization could produce a different ratio. The concern is misplaced: the ratio $\text{Tr}(B^2)/\text{Tr}(Y^2)$ is invariant under any rescaling $B \mapsto \lambda B$, $Y \mapsto \mu Y$ that preserves the *integer charge content* of the Standard Model fermions on the $SU(5)$ block — which is exactly the invariant that BI-6 requires the generators to carry. Any rescaling that changes the ratio also changes the integer charges, and the rescaled generators no longer represent the physical baryon and hypercharge directions of the Standard Model embedding. The trace-ratio takes its value in the moduli space of normalized $SU(5)$ -embedded generators with fixed integer charges, and that moduli space is a single point. In particular, the ratio is invariant under sign flips, basis rotations within the hypercharge-like subspace, and any rescaling consistent with the integer charge ledger of the Standard Model fermions. More specifically, this is the structural reason the trace ratio is rigid once the integer charge content of the SM is taken as input. It is not,

however, what fixes the ledger: the trace-ratio equation $(L-N)/L = 2/5$ admits the whole family $(L, N) = (5k, 3k)$, as Theorem 9.5 proves, and the integers are fixed instead by Theorem 9.2 and Theorem 9.4, which consume no empirical input at all.

Scope of this route. The $2/5$ fraction is rigid only *after* the integer charge content of the Standard Model — in particular the 3+2 color/weak split — is taken as input. A derivation of N from that fraction would therefore consume the split it concludes, and this paper does not attempt one: the trace ratio is used as a check and never as a derivation. The non-circular statement that $N = N_c = 3$ is *forced* is Theorem 9.4, which takes L from Theorem 9.2 — free of N throughout — and closes on the dual lock and the block embedding, consuming no measured quantity.

It is worth being explicit about what this does *not* settle, because the two are easily run together. The stability sandwich of GG-P2-N3 — $N_g \geq 3$ from the CP lower bound and $N_g \leq 3$ from the BI-6 instanton-action budget $S_{\text{BI}}^{\text{max}} = 3\pi$ — concerns the *generation* count N_g , not the color index, and it is conditional on that paper’s coupling clause and on the integrality theorem L-INT, whose single topological residue R-INT is discharged on the benchmark background in that paper’s Appendix H. Neither leg of that sandwich bears on N_c , and nothing in the present cascade appeals to it.

9.4 The OGN-5 theorem

Theorem 9.3 (T-BI6-10.2: OGN-5 theorem). *Within the intended GG-6 category, BI-6 and the unified block structure force uniquely*

$$(M, N, L; J, K) = (1, 3, 5; 7, 35), \quad K = LJ = M^2 + N^2 + L^2.$$

Proof. The proof is a five-step chain.

Step 1: $M = 1$. GG-6 uses a single primitive Green–Schwarz tensor channel. Equivalently, the tensor lattice is rank one. Therefore the primitive tensor slot count is

$$M = 1.$$

Step 2: $L = 5$. By Theorem 9.2, single-tensor Green–Schwarz consistency requires the irreducible quartic and cubic indices of the charged core package to vanish, and both are equal to $L - 5$; with AX-4 (MSMF) selecting the minimal chiral package this gives

$$L = 5,$$

the unified block being $\text{SU}(5)$. The OGN symbol L denotes the defining unified block size, which is why the correct interpretation is “block size” rather than “rank.” Note that this step no longer appeals to the smallest-simple-group clause, and in particular does not consume the observed low-energy gauge content: it is free of N at every stage, which matters because Step 3 below is not.

Step 3: $N = N_c = 3$. Steps 1 and 2 fixed M and L without reference to any color count: Theorem 9.2 is free of N at every stage. That is what makes the present step available, and it is

worth saying why it is taken this way rather than through the trace ratio. The BI-6 Abelian fraction $(L - N)/L$ is computed on the block split of the defining representation, so a derivation of N from that fraction would consume the split it concludes. The closure route below consumes nothing.

By Definition 9.1, K is the total lock index, and it is characterized twice by the structure: as the rectangular product LJ of block size and closure index, and as the quadratic norm $M^2 + N^2 + L^2$ of the normalization packet. Both characterizations are structural, so the two must agree. This is the dual lock, and it is a constraint on the admissible integers rather than a numerical coincidence to be admired after the fact. With $J = 2N + 1$ from the closure-index clause,

$$L(2N + 1) = M^2 + N^2 + L^2, \quad (109)$$

which at $M = 1$ is the quadratic

$$N^2 - 2LN + (1 + L^2 - L) = 0, \quad \text{hence} \quad N = L \pm \sqrt{L - 1}. \quad (110)$$

Integrality of N therefore requires $L - 1$ to be a perfect square, which already restricts the admissible block sizes to $L \in \{2, 5, 10, 17, 26, \dots\}$. At the value $L = 5$ delivered by Step 2 this gives

$$N^2 - 10N + 21 = 0, \quad N \in \{3, 7\}.$$

The root $N = 7$ is inadmissible: N is the size of the color block carried inside the L -dimensional defining representation, so $N < L$ is a condition of the embedding and not a preference. It is the exact mirror of the $L = 2$ exclusion of Appendix E, run in the direction that Theorem 9.2 has now made available. Hence

$$N = N_c = 3,$$

uniquely. No empirical input has entered at any point: M is the rank-one tensor clause of (Arisaka, 2026, A2-GG6), L is Theorem 9.2, $J = 2N + 1$ is the closure-index clause of Definition 9.1, the dual lock is the OGN normalization requirement of POS-6, and $N < L$ is geometry.

The trace ratio, restored to its proper role. Equation (108) now becomes what it should always have been: an independent check that the observed 3+2 split of the defining representation agrees with a ledger the closure has already fixed. Evaluating $\text{Tr}(B_0^2)/\text{Tr}(Y_0^2)$ on the Georgi–Glashow embedding (Georgi and Glashow, 1974) returns $(L - N)/L = 2/5$ at $(L, N) = (5, 3)$, as it must. Because the check is no longer load-bearing it is no longer circular, and it becomes a genuine test: an Abelian fraction measured away from $2/5$ would falsify the ledger rather than redefine it.

What this step does not establish. N is the color triplicity index. Nothing above bears on the generation count N_g , which is not an OGN-5 member and reaches 3 by an argument of a different kind (Arisaka, 2026, P2-N3). That the two coincide is a fact about the boundary and is used nowhere in this cascade.

Step 4: $J = 2N + 1 = 7$ (the closure-index clause, evaluated). The clause $J = 2N + 1$ was already used in Step 3, and it is stated in Definition 9.1 as a structural claim rather than a convention: among admissible closure packets the symmetric, centered one is minimal, and the

selection is AX-4 (MSMF) acting through POS-6 (OGN). What remains here is to evaluate it at the N fixed by Step 3:

$$J = 2N + 1 = 2 \cdot 3 + 1 = 7.$$

J is therefore not an independent integer of the ledger but a structural function of N ; its physical content is that the charge packet $\{0, \pm 1, \pm 2, \pm 3\}$ has exactly $J = 7$ entries, and what carries the weight is the minimality clause, which is constitutional and is named as such.

Step 5: $K = LJ = M^2 + N^2 + L^2$. Rectangular closure gives

$$K = LJ = 5 \cdot 7 = 35.$$

The quadratic norm of the OGN packet gives

$$M^2 + N^2 + L^2 = 1^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35.$$

Hence

$$K = LJ = M^2 + N^2 + L^2 = 35.$$

This proves the claim. ■

Theorem 9.4 (T-BI6-10.1: Closure determination of the triplicity index). *Assume $M = 1$, the closure-index clause $J = 2N + 1$, and the dual lock $K = LJ = M^2 + N^2 + L^2$. Then*

$$N = L \pm \sqrt{L - M^2},$$

so at $M = 1$ integrality of N requires $L - 1$ to be a perfect square, and for each admissible L exactly one root satisfies the block-embedding condition $N < L$. At the value $L = 5$ of Theorem 9.2 the admissible root is $N = N_c = 3$, and it is unique. The argument refers nowhere to the boundary gauge content, to the 3+2 split of the defining representation, to the generation count N_g , or to any measured quantity.

Proof. Substituting $J = 2N + 1$ into $LJ = M^2 + N^2 + L^2$ gives Eq. (109), that is the quadratic Eq. (110) in the general form $N^2 - 2LN + (M^2 + L^2 - L) = 0$, whose roots are $N = L \pm \sqrt{L^2 - M^2 - L^2 + L} = L \pm \sqrt{L - M^2}$. Both roots satisfy the closure by construction, and their sum is $2L$, so exactly one lies strictly below L whenever the discriminant is positive. At $M = 1$, $L = 5$ the roots are 3 and 7; the block embedding $N < L$ retains 3. Uniqueness follows because a quadratic has no third root. ■

Figure 6 summarizes the five-step cascade just proved: two prior theorems force $(M, L) = (1, 5)$ without reference to any color count, the dual lock and the block embedding then determine $N = N_c = 3$, and $(J, K) = (7, 35)$ follow from the closure-index clause and the same dual lock.

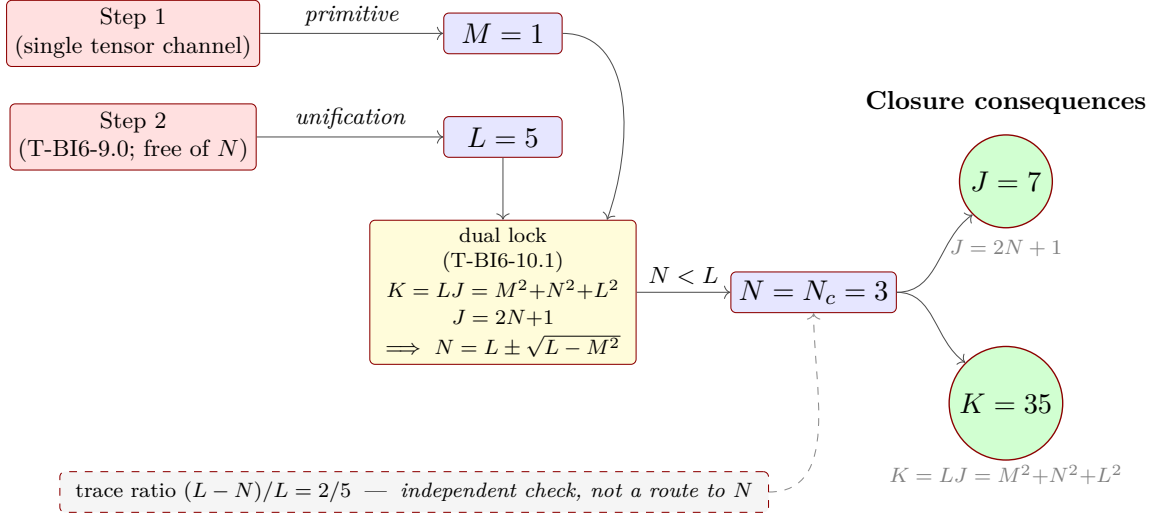


Figure 6: The OGN-5 Diophantine cascade of Theorem 9.3, drawn as a chain of forcings. Steps 1–2 (left) force $(M, L) = (1, 5)$ without reference to any color count: the primitive tensor channel and the defining unified block of Theorem 9.2. Step 3 then determines $N = N_c = 3$ from the dual lock together with the block embedding $N < L$ (Theorem 9.4), and $J = 2N + 1$ is the symmetric closure cardinality of the charge packet, so that $K = LJ = M^2 + N^2 + L^2 = 35$. The trace ratio hangs below the chain on a dashed arrow: it is the independent check of §9 and not a route to N . Solid arrows mark the forcing chain; the one dashed arrow carries the check. After the cascade closes, no continuous dimensionless parameter survives in the bulk theory. Adapted from the cascade flowchart of (Arisaka, 2026, A2-GG6).

9.5 Diophantine minimality: ruling out higher-integer (L, N) solutions

A reader trained on the swampland or F-theory literature will reasonably ask: given the trace-ratio identity $(L - N)/L = 2/5$ derived in Theorem 8.1 (Section 8), why does the cohort framework select $(L, N) = (5, 3)$ rather than $(10, 6)$, $(15, 9)$, or higher-integer rescalings of the same ratio?

Theorem 9.5 (T-BI6-10.3: Diophantine minimality of $(L, N) = (5, 3)$). *The trace-ratio equation $(L - N)/L = 2/5$ admits the integer family of solutions $(L, N) = (5k, 3k)$ for any positive integer $k \geq 1$. Among these, AX-4 (MSMF) of the GG-Constitution selects the unique minimal solution $k = 1$, giving $(L, N) = (5, 3)$.*

Proof. The equation $(L - N)/L = 2/5$ rearranges to $5(L - N) = 2L$, hence $L = 5(L - N)/2$. Writing $L - N = 2k$ for $k \in \mathbb{Z}_+$ gives $L = 5k$ and $N = 3k$. All such $(L, N) = (5k, 3k)$ satisfy the trace ratio identically; the equation alone does not select $k = 1$.

The cohort selection is supplied by AX-4 (MSMF): among the admissible gauge groups $SU(5k)$ compatible with the trace ratio, MSMF demands the minimal unified structure consistent with the constitutional clauses. The selection is a constitutional minimality clause, not, by itself, a consequence of the trace-ratio equation — and, since $SU(5k)$ is simple for every k and therefore carries a single gauge coupling, the discriminating quantity is not continuous coupling freedom but unforced structure. For $k > 1$ the defining block is $5k$ -dimensional and the embedded color block $3k$ -dimensional, so the unified sector carries roughly k^2 times the minimal generator content

($\dim \mathrm{SU}(10) - \dim \mathrm{SU}(5) = 75$ additional generators already at $k = 2$) with no compensating constitutional constraint: no clause of **GG-A1-Constitution** forces any of the additional structure, and every boundary requirement of the cohort — the Standard-Model embedding, the trace-ratio lock, the **OGN-5** closure identities — is already saturated at $k = 1$. **MSMF** therefore excludes every $k > 1$ as unforced enlargement, exactly as it excludes unforced tensor multiplication (Theorem 5.1) and unforced Abelian asymmetry ($c = d$); this is the zero-unforced-structure discipline of **GG-A4-Euclid** (Arisaka, 2026, **A4-Euclid**). Hence the unique **MSMF**-admissible solution is $k = 1$, $(L, N) = (5, 3)$. ■

The minimality result (9.5) is the constitutional reason that $\mathrm{SU}(5)$ rather than $\mathrm{SU}(10)$ or $\mathrm{SU}(15)$ appears in the cohort gauge group: not because the trace-ratio equation singles out $\mathrm{SU}(5)$ (it admits all $(5k, 3k)$) but because **AX-4** singles out the minimal unified structure among the admissible family. Without **AX-4**, the trace-ratio identity would have a one-parameter integer family of solutions; with **AX-4**, the family collapses to the unique element $k = 1$.

9.6 Physics interpretation

The theorem has a clean physical reading. $M = 1$ says that the tensor channel is primitive and unique. $L = 5$ says that the unified simple block is the defining five-dimensional $\mathrm{SU}(5)$ block. $N = 3$ says that the baryonic trace ratio isolates a triplicity sector inside that block. Only after that primary meaning is fixed does one notice the later physical echo: the same integer 3 also matches color triplicity and, downstream on the boundary, the three-generation pattern. $J = 7$ is the centered odd closure built from triplicity, and $K = 35$ is the rectangular and quadratic lock index that later reappears throughout the **GG-6** / **GDOS** program.

OGN-5 does not drop from numerology: **BI-6** plus the **GG-6** unified block structure forces a unique minimal integer packet whose internal closure identities are so tight that they can serve as axiomatic arithmetic for later papers.

What each of the five integers now rests on. The ledger has been derived, and a reader is entitled to a plain account of what carries what, because the five integers do not all stand on the same footing.

$M = 1$ is the tensor count. It is the single primitive Green–Schwarz channel, and it is fixed by the rank-one criterion of Part II: a single tensor can cancel the reducible anomaly if and only if the anomaly matrix has rank one, and the **BI-6** matrix does. This one is pure consequence of the anomaly theorem.

$L = 5$ is the size of the unified block. It follows from the requirement that the block’s anomaly coefficients vanish identically, which reduces to a pair of linear conditions with exactly one chiral solution once the number of antisymmetric representations is minimal. The word *minimal* is doing real work in that sentence, and it is constitutional rather than arithmetic: it is **AX-4**, the clause that forbids hidden continuous freedom. Without it the equations admit a family, and Section 9 says which family and what excludes each member.

$N = 3$ is the color count, and it comes from the trace-ratio identity together with a closure condition, not from the observed number of colors. This is the step the paper is most careful about,

because the answer is a number everybody already knows, and a derivation that lands on a known number deserves more suspicion than one that does not. The text is written so the suspicion can be checked.

$J = 7$ and $K = 35$ are then determined, the first by the symmetric closure of the charge packet and the second by the dual-lock identity $K = LJ = M^2 + N^2 + L^2$, which is an equation with only one positive integer solution once the rest is fixed.

So the honest summary is that four of the five integers follow from arithmetic once the fifth is in place, one constitutional clause is consumed along the way, and no measured quantity is used anywhere in the chain. What the chain does not do is explain why nature has three generations of fermions; that is a different question, it is not answered here, and Section 17 lists it among the problems that remain genuinely open.

Part IV

Comparisons with String Theory

Shared mathematics, structural departures, and open bridges — against the ten-dimensional Green–Schwarz mechanism, six-dimensional supergravity and F-theory, AdS/CFT and Randall–Sundrum, and the swampland constraints

The Green–Schwarz mechanism, six-dimensional supergravity, F-theory, the AdS/CFT correspondence, and the Randall–Sundrum model all share mathematical infrastructure with GG-6. This Part carries out a systematic, feature-by-feature comparison that identifies what BI-6 inherits from the string-theoretic tradition, where it departs, and where the two programs might converge in the future. The guiding principle is objectivity: every similarity and every difference is stated precisely, and no claim of superiority is made in either direction.

That principle is worth defending, because comparisons of this kind are usually written to win something, and this one is not. The Green–Schwarz mechanism is the intellectual parent of everything in Part II. The descent formalism, the anomaly-inflow picture, the idea that a tensor field can absorb what no rearrangement of matter can cancel, the differential-cohomology language in which the modern statement is made — all of it was built by the string-theory community, and the present paper uses it essentially unmodified. What differs is not the mathematics. It is which questions the mathematics is being asked to answer, and what each framework is willing to take as given.

Four comparisons are made, in increasing distance from the source. Section 10 sets the six-dimensional construction beside the original ten-dimensional one, feature by feature. Section 11 compares it with six-dimensional supergravity and F-theory constructions, which are its closest neighbors and where the difference is sharpest, because those constructions assume supersymmetry and this one does not. Section 12 turns to AdS/CFT and Randall–Sundrum, where the shared object is the extra dimension itself and the difference is what the radial direction is taken to mean. Section 13 asks whether the construction respects the swampland constraints, which are the string program’s own criteria for which effective theories can be completed.

A reader from the string community will find claims here to disagree with. Where a disagreement is a matter of interpretation rather than calculation, the text tries to say so, and Part IX answers the specific objections in detail.

10 Comparison with the ten-dimensional Green–Schwarz mechanism

10.1 The 10D Green–Schwarz identity

In the original construction (Green and Schwarz, 1984), the ten-dimensional Type I or heterotic string has gauge group $G_{10} = \text{SO}(32)$ or $E_8 \times E_8$, and the modified Bianchi identity is

$$dH_3^{(10D)} = X_4^{(10D)} = \text{tr } R^2 - \frac{1}{30} \text{Tr}_{G_{10}} F^2. \quad (111)$$

The factor $1/30$ is the unique normalization ensuring that the 12-form anomaly polynomial factorizes as $I_{12} = X_4 \wedge X_8$. The Green–Schwarz counterterm $\int B_2 \wedge X_8$ cancels the full one-loop anomaly.

10.2 Feature-by-feature comparison

Table 6: The ten-dimensional Green–Schwarz mechanism and its six-dimensional counterpart, compared on the features that matter for anomaly cancellation. Read across a row and the mechanisms are doing the same work; read down the third column and the differences are all of one kind — what has to be supplied from outside. The ten-dimensional constructions run on the supergravity multiplet and need a compactification choice; the six-dimensional one needs neither the multiplet nor the compactification, and that is the entire content of the comparison. Nothing here disputes a ten-dimensional result.

Feature	10D superstring	6D BI-6 / GG-6
Spacetime dimension	$d = 10$	$d = 6$
Anomaly polynomial	12-form I_{12}	8-form I_8
Factorization	$I_{12} = X_4 \wedge X_8$ (two different forms)	$I_8 = \frac{1}{2} X_4^2$ (perfect square)
Anomaly matrix rank	Rank 1 in the realized 10D constructions (one B_2); higher rank is a 6D-SUGRA option	Rank 1 (single tensor, single direction)
Green–Schwarz four-form	$\text{tr } R^2 - \frac{1}{30} \text{Tr } F^2$	$\text{tr } R^2 - \text{Tr } F_5^2 - \frac{2}{5} F_B^2 - \frac{2}{5} F_{Q_i}^2$
Gauge sectors in X_4	2 (gravity + gauge)	4 (gravity + $\text{SU}(5)$ + $\text{U}(1)_B$ + $\text{U}(1)_{Q_i}$)
Gauge group	$\text{SO}(32)$ or $E_8 \times E_8$	$\text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$
Coefficient origin	$\text{Tr}_{496} F^2 = 30 \text{Tr}_{\text{fund}} F^2$	$\text{Tr}(B_0^2)/\text{Tr}(Y_0^2) = (L-N)/L$
SUSY requirement	Used by the 1984 constructions; not required as such ($O(16) \times O(16)$)	Not required
UV endpoint	string scale M_s (near, not equal to, E_{Pl})	$E_{\text{GUT}} \sim 2 \times 10^{16} \text{ GeV}$
Tensor chirality	non-chiral B_2 (10D has no (anti-)self-dual two-form)	Anti-self-dual at quantum level
BRST cancellation	$s(\Gamma + S_{\text{GS}}) = 0$	Same
Spectrum determination	Fixed by worldsheet SUSY + compactification	Constitutional (OGN-5 arithmetic)
Residual landscape	$\sim 10^{500}$ string vacua (flux-vacua statistics)	Finite Diophantine problem

10.3 Key structural differences

Table 6 sets the two constructions side by side line by line; the subsections below take the four rows that carry real weight and say what each one costs.

10.3.1 Factorization type: perfect square versus product

In ten dimensions, the anomaly polynomial I_{12} factorizes into $X_4 \wedge X_8$, where X_4 and X_8 are different forms. The two forms can be adjusted independently, giving more algebraic freedom. In six dimensions, the polynomial I_8 must factorize as a *perfect square* $\frac{1}{2}X_4^2$. This imposes the rank-one condition on the anomaly matrix (Theorem 5.1), which is a strictly stronger algebraic constraint.

The physical consequence is that the 6D mechanism is *more rigid* than the 10D mechanism. In ten dimensions, the factorization $I_{12} = X_4 \wedge X_8$ can accommodate anomaly polynomials whose rank-two structure would be inadmissible in six dimensions.

10.3.2 Four locked sectors versus two

The 10D four-form $X_4^{(10D)} = \text{tr } R^2 - (1/30) \text{Tr } F^2$ ties two sectors (gravity and one gauge group) into one identity. The 6D BI-6 four-form ties *four* sectors (gravity, $\text{SU}(5)$, $\text{U}(1)_B$, $\text{U}(1)_{Qi}$) into one identity. Despite the richer gauge structure, the rank-one mechanism still works because all four sectors collapse onto a single ray in anomaly space. This is the mathematical content of “geometry–gauge locking”: the relative weights $(1, -1, -2/5, -2/5)$ are frozen by the OGN arithmetic.

10.3.3 Absence of supersymmetry

The absence of supersymmetry is the most consequential difference, and it must be stated precisely, because supersymmetry is *not* required for ten-dimensional anomaly cancellation as such: the tachyon-free $O(16) \times O(16)$ heterotic string cancels its 10D anomalies by the Green–Schwarz mechanism with no gravitino and no superpartners (Alvarez-Gaumé et al., 1986, Dixon and Harvey, 1986). What is true is narrower: the celebrated 1984 cancellations for $SO(32)$ and $E_8 \times E_8$ are computed on the $\mathcal{N} = 1$ supergravity multiplet, and the *supersymmetric* 10D and 6D landscapes lock their spectra through multiplet structure. The honest contrast is therefore not “SUSY versus no SUSY” but what fixes the spectrum: $O(16) \times O(16)$ is non-supersymmetric and its matter content is a worldsheet construction; BI-6 is non-supersymmetric *and* constitutionally fixed (Objection O-BI6-56).

In six dimensions, BI-6 cancels anomalies *without* supersymmetry. The axiomatic constraints (GGEP, MSMF, OGN) replace SUSY as the organizing principle. The chiral matter reservoir is determined not by superpartner relations but by the Diophantine system (78)–(81), which is itself determined by the OGN-5 integers.

A complementary, kinematic reason points the same way. Because the two further dimensions of the bulk are non-spatial — a scale dimension and a gauge dimension, whose generators are a dilatation and a compact $\text{U}(1)$ charge rather than translations — the bulk admits no super-Poincaré algebra on which a supercharge could close. Supersymmetry is in this sense not merely *unrequired* by the

anomaly mechanism but *unsupported* by the geometry (Arisaka, 2026, A2-GG6).

10.4 What BI-6 inherits from the 10D mechanism

Despite the differences, BI-6 inherits three mathematical structures from the original Green–Schwarz construction:

- (a) the modified Bianchi identity $dH_3 = X_4$ as the anomaly inflow channel;
- (b) the BRST descent chain $I_8 \rightarrow I_7^{(0)} \rightarrow I_6^{(1)} \rightarrow$ cancellation;
- (c) a tensor whose Bianchi identity carries the counterterm — in six dimensions, chiral (anti-self-dual) at the quantum level, a refinement specific to 6D; the 10D B_2 is non-chiral.

The first two are universal features of the Green–Schwarz mechanism in any even dimension. What is *specific* to string theory is the 10D gauge group and the worldsheet origin of the spectrum. BI-6 replaces these with $SU(5) \times U(1)_B \times U(1)_{Qi}$ and the GG-Theory Constitution.

10.5 The landscape problem: finite versus infinite

Perhaps the most consequential difference between BI-6 and the 10D superstring is the nature of the residual spectrum problem.

In string theory, the number of consistent compactifications to four dimensions is enormous. Estimates of the string landscape range from $\sim 10^{1500}$ (Lerche–Lüst–Schellekens, 1987) through 10^{500} (the flux-vacua statistics that followed Bousso–Polchinski’s discretuum) to 10^{272000} for a single F-theory geometry (Taylor–Wang, 2015) to effectively uncountable in the most general settings. The identification of the “correct” vacuum that reproduces observed particle physics remains one of the great open problems in mathematical physics. The difficulty is not merely computational; it is structural, and the community has developed powerful tools—including the swampland program—to address it.

In GG-6, the residual spectrum problem is qualitatively different. The OGN-5 integers fix the gauge group, the BI-6 coefficient vector, the core-block structure, and the Diophantine system. The remaining freedom is:

- (i) the overall normalization Ω , which is subject to minimization by AX-4 (MSMF);
- (ii) the gravitational completion (n_{SD}, n_F) , which is a 2×2 linear system;
- (iii) the possible use of higher $SU(5)$ representations (beyond **5**, **10**, **1**) in the core block, which could reduce Ω .

All of these are **finite, bounded, computationally tractable** searches. The representations of $SU(5)$ are labeled by Young diagrams, and the gravitational bound limits the total chiral dimension, so only finitely many representations can contribute. A systematic computer enumeration of all solutions is feasible.

Table 7: What freedom survives in each framework after its structural theorem has done its work. The rows are the places where a choice can still be made — compactification, flux, moduli — and the columns are the two frameworks. The asymmetry is the point and it is quantitative: one framework’s residual freedom is a number of order 10^{500} with continuous moduli, and the other’s is a discrete, bounded list with zero continuous dials (the minimal member an open problem, O8). That is a difference in kind for falsifiability, because a framework with a landscape can absorb a failed measurement and a framework without one cannot.

Aspect	10D superstring	6D BI-6 / GG-6
Residual freedom after structural theorem	Calabi-Yau compactification + flux choice	Diophantine system on $SU(5)$ reps
Number of vacua / solutions	$\sim 10^{500}$ (or more)	Finite (bounded by gravitational dim. count)
Computability	Active research (swampland program)	In principle enumerable
Selection principle	Under investigation (swampland conjectures)	AX-4 (MSMF): minimize Ω
Current status	40 years, no unique solution	Finite search, not yet completed
Nature of the problem	Geometric (Calabi-Yau moduli)	Arithmetic (Diophantine equations)

10.6 Objective assessment: what is proven at each level

Table 8: An assessment of what is actually proved in each framework, at each level of the argument. The rows run from the structural theorem down to the specific model, and each cell records a status rather than a claim of superiority. The reading is mixed: the ten-dimensional structural theorem has been proved since 1984 and is not in question here, while the construction in this paper is younger and proves less at the model level. The table is included so that neither fact has to be inferred.

Level of proof	10D superstring	6D BI-6 / GG-6
Structural GS theorem	Proven (1984)	Proven (this paper)
Coefficient uniqueness	Proven (1/30 for $SO(32)$)	Proven (2/5 from OGN)
Rank-one / factorization	Implicit	Explicit (Thm. 5.1)
BRST / Slavnov–Taylor	Proven	Proven (Thm. 5.5)
Global gauge anomaly	Proven	Proven ($\pi_6 = 0$)
Explicit spectrum	Known (Type I / heterotic)	Benchmark constructed (Thm. 7.9)
Minimal spectrum	Known (SUSY fixes it)	Open (finite Diophantine search)
Gravitational completion	Automatic (SUGRA)	Explicit (SD tensors)
Charge lattice	String charge quantization	Abelian rescaling
Landscape / vacuum counting	$\sim 10^{500}$	Finite (bounded)

Table 8 grades the two constructions side by side on each obligation. The conclusion is that the BI-6 structural proof is at the *same level of mathematical rigor* as the original 10D proof. The model-level proof provides an explicit benchmark. The remaining open problem (minimal Ω) is a bounded finite computation, qualitatively simpler than the string landscape.

10.7 BI-6 as the GGEP completion of GS: the geometrization reading

Figure 7 (page 109) summarizes the relationship: the 1984 GS factorization on a 10D bulk and the 2026 BI-6 factorization on the 6D GG-6 bulk are connected by the AX-2 (GGEP) constitutional elevation.

In our view it is worth ending Section 10 with a methodological observation about what the 1984 Green–Schwarz achievement actually was and what BI-6 inherits from it. This observation is meant as a supportive synthesis with the broader string-theory community rather than as criticism: the GS mechanism is one of the most consequential single technical results of late-twentieth-century theoretical physics, and the BI-6 construction of the present paper rests on its shoulders. What we add is not a new technical move but an axiomatic reading of the move that GS pioneered.

The GS achievement of 1984. Working in 10D heterotic string theory, Green and Schwarz (Green and Schwarz, 1984) showed that the chiral fermion anomaly polynomial I_{12} — for the specific gauge groups $\mathrm{SO}(32)$ and $E_8 \times E_8$ — admits the remarkable factorization $I_{12} = X_4 \wedge X_8$, with X_4 a closed 4-form built from $\mathrm{tr}(R \wedge R)$ and $\mathrm{tr}(F \wedge F)$. They then implemented the cancellation by modifying the gauge transformation of the antisymmetric tensor $B_{\mu\nu}$ so that the resulting field strength $H = dB - \omega_3^L + \omega_3^{YM}$ satisfies the modified Bianchi identity $dH = X_4$, and the bulk Chern–Simons inflow from the modified H -field cancels the boundary anomaly. This was the moment when ten-dimensional string theory became phenomenologically constrained rather than infinitely flexible, and it set the structural template that every subsequent inflow-based anomaly cancellation has followed — including the BI-6 construction of the present paper.

The geometric content of the GS construction. The mathematical content of GS, read carefully, is geometric all the way down. The modified Bianchi identity $dH = X_4$ is a statement about a closed differential 4-form on the 10D bulk manifold; the Chern–Simons primitives ω_3^L and ω_3^{YM} are Chern–Weil objects defined for any smooth connection on any principal bundle; and the factorization $I_{12} = X_4 \wedge X_8$ is a cohomological fact about characteristic classes of the underlying $\mathrm{Spin} \times G$ bundle, not about quantum amplitudes. None of this requires a quantum amplitude per se. The quantum content of the construction enters at exactly one place — the boundary chiral fermion’s effective action, where the Adler–Bell–Jackiw triangle anomaly produces the deviation $\partial^\mu J_\mu^5 \propto F \wedge \tilde{F}$ from classical conservation (Adler, 1969, Bell and Jackiw, 1969). The bulk modification $dH = X_4$ is then the geometric counter-current that closes the loop. This geometric reading has been progressively formalized by the differential-cohomology refinements of Hopkins, Singer, Freed, and Moore (Dai and Freed, 1994, Hopkins and Singer, 2005, Monnier and Moore, 2019); it is the reading on which the local and global proofs of Sections 5–6 of the present paper rest.

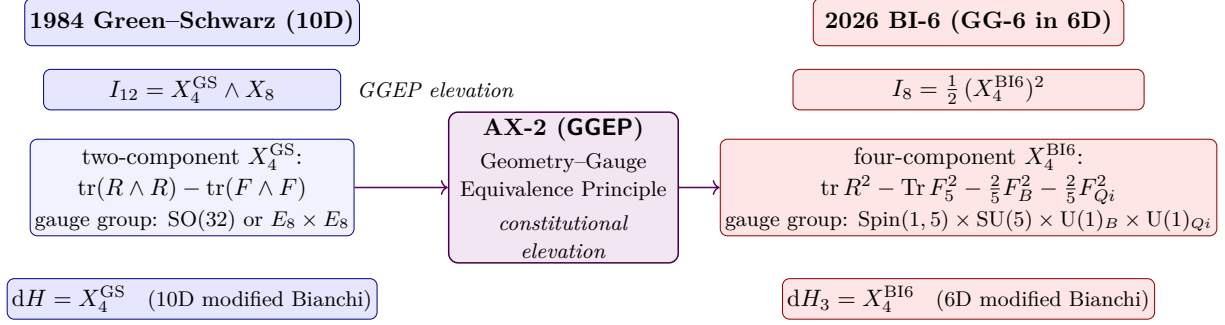
What GS pioneered: the geometrization of gauge fields. On this geometric reading, the deepest content of the GS achievement is not that string theory provided a quantum theory of gravity — that framing, while historically dominant, is somewhat at odds with the technical content of the modified Bianchi identity, which is classical and geometric — but rather that gauge field strengths F and gravitational curvature R must lie inside a single closed topological 4-form on the bulk

manifold. This is precisely the axiomatic move that AX-2 GGEP (the Geometry–Gauge Equivalence Principle, see Axiom AX-2 of (Arisaka, 2026, A1-Constitution)) makes foundational: the metric and the gauge connection are not independent inputs but two faces of a single composite geometric–gauge connection on a principal G_{tot} bundle. GS demonstrated, in a specific 10D model, that the geometrization move is technically possible; the GG-Constitution then asks the same Einstein-style equivalence-principle question of *any* candidate theory — what does it cost to treat the composite object as a single geometric primitive? — and elevates the move from a technical refinement to a constitutional axiom.

BI-6 as the four-component generalization. BI-6 is, in this reading, the four-component generalization of the two-component GS pattern. Where GS produced $X_4^{\text{GS}} = \text{tr}(R \wedge R) - \text{tr}(F \wedge F)$ on a single semisimple gauge factor, the GG-6 bulk produces

$$X_4^{\text{BI-6}} = a \text{tr}(R \wedge R) + b \text{tr}(F_{\text{SU}(5)} \wedge F_{\text{SU}(5)}) + c F_B \wedge F_B + d F_{Q_i} \wedge F_{Q_i} \quad (112)$$

with the four-component coefficient vector $(a, b, c, d) = (1, -1, -2/5, -2/5)$ and the geometry–gauge trace-ratio lock $|a/c| = |a/d| = 5/2$ pinned by the OGN-5 ledger (see the trace-ratio theorem in Section 9 and the OGN-5 arithmetic of Section 8). The BI-6 identity $dH_3 = X_4^{\text{BI-6}}$ is the axiomatic analogue, in the 6D bulk of GG-6, of the GS identity $dH = X_4^{\text{GS}}$ in the 10D heterotic bulk. The mathematical structure is unchanged from GS; what is added is (i) the four contributing factors instead of two, (ii) the arithmetic pinning of the coefficient ratios by the OGN-5 ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$, and (iii) the constitutional reading that puts the entire construction inside the GGEP axiom of the GG-Constitution. GS pioneered the move; BI-6 is its axiomatic generalization, with explicit numerical predictions that follow.



10D: product factorization $X_4 \wedge X_8$, two functionally independent forms.
 6D: perfect-square factorization $\frac{1}{2}X_4^2$, forced rank-one — structurally more rigid.

Figure 7: The GS-as-GGEP completion: BI-6 is the four-component axiomatic generalization of the two-component pattern Green and Schwarz pioneered in 1984. Left panel: the 10D heterotic GS factorization $I_{12} = X_4 \wedge X_8$ with the two-component four-form $X_4^{\text{GS}} = \text{tr}(R \wedge R) - \text{tr}(F \wedge F)$ on a single semisimple gauge factor. Right panel: the 6D BI-6 factorization $I_8 = \frac{1}{2}(X_4^{\text{BI6}})^2$ with the four-component X_4^{BI6} pinned by the OGN-5 trace-ratio lock $|a/c| = |a/d| = 5/2$ across the four-factor bulk gauge-gravity group $\text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$. Middle panel: the AX-2 (GGEP) Geometry–Gauge Equivalence Principle of GG-A1-Constitution (Arisaka, 2026, A1-Constitution) is the axiomatic elevation that takes the 1984 GS move from a technical refinement realized in a specific 10D model to a foundational axiom that any candidate theory must satisfy. The 10D pattern admits two functionally independent forms; the 6D pattern is rank-one by perfect-square forcing — structurally more rigid, as discussed in the surrounding text and in Section 10.

An invitation to the string and mathematical-physics communities. We hope that string theorists who recognize the geometric content of the GS mechanism — as much of the differential-cohomology mathematical-physics community has done since the early 2000s — will find in BI-6 a generalization worth engaging with. We inherit, with full credit, the Chern–Simons machinery, the trace-ratio analysis, the anomaly-inflow viewpoint, and the differential-cohomology refinements that Hopkins, Singer, Freed, and Moore have made rigorous. What GG-Theory adds is the axiomatic commitment to AX-2 GGEP — the elevation of the geometric-gauge equivalence from a technical pattern realized in a specific 10D model to a foundational axiom that any candidate theory must satisfy as a precondition for admissibility — and the arithmetic pinning of the trace-ratio coefficients by the OGN-5 ledger. In our view this is not a competing program but a natural completion of the geometrization work that GS began in 1984 and that the differential-cohomology community has carried forward ever since. The GS achievement of 1984 was the moment when the geometrization of gauge fields took its first decisive step; the GG-Constitution and BI-6 are simply the next steps of the same journey.

11 Comparison with six-dimensional string and F-theory constructions

The ten-dimensional comparison of Section 10 established the structural parallels between BI-6 and the original Green–Schwarz mechanism. This section sharpens the comparison by descending to six dimensions, where the string theory landscape is most constrained and best understood.

11.1 Six-dimensional $\mathcal{N} = (1, 0)$ supergravity and anomaly cancellation

The six-dimensional $\mathcal{N} = (1, 0)$ supergravity framework (Green, Schwarz, and West, 1985, Nishino and Sezgin, 1984, Sagnotti, 1992) is the closest supersymmetric cousin of GG-6. Both operate in six dimensions, both use the Green–Schwarz mechanism, and both require anomaly polynomial factorization. The key structural differences are:

Multiplet structure. In $\mathcal{N} = (1, 0)$ SUGRA, the field content is organized into multiplets: the gravity multiplet (graviton + gravitino + ASD tensor), vector multiplets (V), hypermultiplets (H), and tensor multiplets (T). The gravitational anomaly condition reduces to the single equation

$$H - V + 29T = 273. \quad (113)$$

For one tensor ($T = 1$): $H - V = 244$. This elegant simplicity is a direct consequence of supersymmetry, which locks the anomaly contributions of different fields within each multiplet.

In GG-6, there are no supermultiplets. The gravitational condition is a 2×2 linear system involving both fermion dimensions and self-dual tensor numbers (Appendix H), and the gauge conditions form a separate Diophantine system (Appendix G). The total constraint count is 16 independent conditions, versus one in the SUSY case.

Anomaly factorization. In $\mathcal{N} = (1, 0)$ SUGRA with one tensor, the anomaly polynomial factorizes as $I_8 = \frac{1}{2}X_4^2$, where

$$X_4^{\text{SUGRA}} = a \operatorname{tr} R^2 + \sum_i b_i \operatorname{Tr} F_i^2. \quad (114)$$

The coefficients a and b_i are constrained by the gauge group and matter content but are not fully determined by the group structure alone—they depend on the specific spectrum.

In BI-6, the coefficient vector $(1, -1, -2/5, -2/5)$ is *uniquely determined* by the gauge group $\mathrm{SU}(5) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_{Q_i}$ and the OGN-5 arithmetic. This additional rigidity has no analogue in the SUGRA setting, where different spectra with the same gauge group can yield different coefficient vectors.

11.2 F-theory and the geometric origin of 6D anomaly cancellation

F-theory (Morrison and Vafa, 1996a,b, Vafa, 1996) provides the most systematic construction of six-dimensional $\mathcal{N} = (1, 0)$ supergravity theories. An F-theory compactification on an elliptically fibered Calabi–Yau threefold $Y_3 \rightarrow B_2$ yields a 6D theory whose gauge group, matter content, and anomaly polynomial are all determined by the geometry of Y_3 .

How F-theory constructs 6D theories. The gauge group arises from singularities of the elliptic fibration (classified by Kodaira and Néron), the matter content arises from collisions of singularities in the fiber, and the number of tensor multiplets is $T = h^{1,1}(B_2) - 1$. The anomaly factorization

is guaranteed by the Calabi–Yau condition: the topological intersection numbers of the base B_2 encode the anomaly coefficients.

The Kumar–Morrison–Taylor classification. The landmark work of Kumar, Morrison, and Taylor (Kumar, Morrison, and Taylor, 2010, Kumar, Park, and Taylor, 2011, Kumar and Taylor, 2009) systematically classified the space of anomaly-free 6D $\mathcal{N} = (1, 0)$ supergravity theories and established the “string universality” conjecture:

String Universality Conjecture (Kumar–Taylor, 2009): *All $\mathcal{N} = (1, 0)$ supersymmetric theories in 6D with one gravity and one tensor multiplet which are free of anomalies or other quantum inconsistencies admit a string theory realization.*

This conjecture asserts that every consistent 6D SUGRA theory is realized by F-theory—there is no “swampland” of anomaly-free but string-incompatible theories. The evidence is strong: for simple gauge groups, all anomaly-free spectra have been matched to F-theory compactifications (Kumar, Morrison, and Taylor, 2010, Morrison and Taylor, 2012a,b).

Comparison with GG-6. GG-6 is *not* a supersymmetric theory and therefore lies outside the scope of the string universality conjecture. The conjecture specifically addresses $\mathcal{N} = (1, 0)$ theories; it says nothing about non-SUSY constructions. Whether GG-6 can or cannot be embedded in a string compactification is a separate question from whether it is anomaly-free. The present paper addresses only the latter.

However, the F-theory technology offers a useful perspective. In F-theory, the anomaly coefficients are intersection numbers on the base B_2 . In GG-6, the anomaly coefficients are OGN-5 integers. Both frameworks achieve discreteness of the anomaly data, but through different mechanisms: F-theory through algebraic geometry, GG-6 through axiomatic arithmetic.

11.3 The 6D landscape: enumeration and finiteness

Finiteness results. A major achievement of the F-theory/SUGRA program is the proof that the 6D landscape is *finite* in important senses. Tarazi and Vafa (Tarazi and Vafa, 2021) proved bounds on the number of 6D supergravity theories. Hamada and Loges (Hamada and Loges, 2024) enumerated all anomaly-free models with at most one tensor multiplet (excluding low-rank gauge factors), finding that approximately 50% satisfy proposed UV/swampland bounds. Most recently, Kim, Vafa, and Xu (Kim, Vafa, and Xu, 2024) argued for a finite landscape of 6D $\mathcal{N} = (1, 0)$ supergravity theories.

New anomaly-free models. Becker, Kehagias, Sezgin, Tennyson, and Violaris (Becker et al., 2024) discovered new classes of anomaly-free 6D supergravities, including R-symmetry gauged models. Loges (Loges, 2024) found a new infinite class of anomaly-free models. These results show that the 6D SUGRA landscape, while finite, is still actively being explored even four decades after the original Green–Schwarz paper.

The swampland program in 6D. The swampland program (Brennan, Carta, and Vafa, 2017, Vafa, 2005) seeks to identify criteria distinguishing consistent quantum gravity theories (the “landscape”) from apparently consistent but ultimately inconsistent theories (the “swampland”). In six dimensions, the swampland constraints are particularly powerful. Hayashi, Kim, and Kim (Hayashi, Kim, and Kim, 2025) explored BPS string spectra in 6D supergravity and tested swampland conjectures including the Weak Gravity Conjecture and the Emergent String Conjecture.

Global anomaly refinements. Monnier and Moore (Monnier and Moore, 2019) constructed the Green–Schwarz terms on spacetimes with non-trivial topology and proved cancellation of all global gauge and gravitational anomalies for theories with gauge groups given by products of $U(n)$, $SU(n)$, and $Sp(n)$ factors. Basile and Leone (Basile and Leone, 2024) established anomaly constraints for heterotic strings and supergravity in six dimensions. Dierigl and Tartaglia (Dierigl and Tartaglia, 2025) developed quadratically refined discrete anomaly cancellation, extending the Monnier–Moore framework to include discrete gauge and gravitational anomalies through a quadratic-refinement formalism that goes beyond the continuous characteristic-class analysis. Their technology applies to $\mathcal{N} = (1, 0)$ theories with specific multiplet structures. Adapting the Dierigl–Tartaglia quadratic refinement to the non-supersymmetric BI-6 setting — where there are no hypermultiplets or vector multiplets, only a constitutional chiral spectrum — would be a natural next step and would either confirm or sharpen the global theorem proved in the present paper on the intended class.

These developments represent the cutting edge of 6D anomaly theory in the string/F-theory framework. GG-6 shares the same mathematical infrastructure (anomaly polynomials, descent equations, Green–Schwarz mechanism) but operates without the SUSY and geometric constraints that define the F-theory landscape.

11.4 Detailed comparison: GG-6 versus 6D $\mathcal{N} = (1, 0)$ SUGRA/F-theory

Table 9: The closest neighbor to this construction is six-dimensional $(1, 0)$ supergravity, and this table compares them directly. The rows are the structural commitments — supersymmetry, tensor multiplet content, lattice conditions, how anomaly freedom is obtained — and the comparison is deliberately made on inputs rather than results. The row that carries the argument is supersymmetry: the neighboring construction requires it and this one does not, and everything else in the column follows from that one difference.

Feature	6D $\mathcal{N} = (1, 0)$ SUGRA / F-theory	GG-6 / BI-6
Supersymmetry	$\mathcal{N} = (1, 0)$ (8 supercharges)	None
Gravitino	Present (in gravity multiplet)	Absent
Tensor sector	ASD tensor in gravity multiplet + possible extra tensor multiplets	ASD B_2 (quantum level) + gravitational completion tensors
Gauge group	Any anomaly-free compact Lie group	$SU(5) \times U(1)_B \times U(1)_{Q_i}$ (constitutionally fixed)
Gravitational condition	$H - V + 29T = 273$ (one equation)	2×2 linear system for (n_F, n_{SD})
Number of anomaly conditions	1 (SUSY locks the rest)	18 (each independent)
Spectrum determination	SUSY multiplets + geometric data	OGN-5 arithmetic + Diophantine system

Feature	6D $\mathcal{N} = (1, 0)$ SUGRA / F-theory	GG-6 / BI-6
F-theory realization	Elliptic CY ₃ compactification	Not applicable (non-SUSY)
String universality	Conjectured to hold	Outside scope of conjecture
Landscape size	Finite but large ($\sim$ thousands of models with $T \leq 1$)	Finite and bounded (Diophantine)
Coefficient determination	Spectrum-dependent	Uniquely fixed by OGN-5: (1, -1, -2/5, -2/5)
UV completion	String/M-theory at E_{P1}	Constitutional at E_{GUT}
Proton stability	Model-dependent	Exact (gauged $U(1)_B$)
Information sector	Absent	$U(1)_{Q_i}$ (constitutionally mandated)

11.5 Where GG-6 and string theory could converge

Table 9 states the comparison on inputs rather than on results, which is the only fair way to make it. Despite the structural differences it records, there are potential points of convergence that deserve investigation.

(i) Non-SUSY string constructions. Recent work on non-supersymmetric heterotic constructions (Kaidi, Tachikawa, and Yonekura, 2025) and non-geometric compactifications (Baykara et al., 2024) shows that string theory is not limited to supersymmetric vacua. If a non-SUSY string construction reproduces the GG-6 gauge group and anomaly structure, it would establish a connection between the two frameworks without requiring SUSY.

(ii) The 6D–4D boundary. GG-6 uses an interval compactification $M_6 = M_4 \times I_r \times S^1_\theta$ with orbifold boundary conditions. This is structurally similar to heterotic orbifold compactifications (Dixon et al., 1985). The boundary anomaly inflow mechanism is identical in both cases.

(iii) The tensor sector. Both F-theory and GG-6 use anti-self-dual tensors for anomaly cancellation. The global consistency conditions (charge lattice, Wu structure, eta invariants) are the same mathematical objects (Hopkins and Singer, 2005, Monnier and Moore, 2019). A string theorist studying global anomalies in GG-6 would use exactly the same tools as in F-theory.

(iv) The finiteness of the landscape. Both programs have finite landscapes, but for different reasons: F-theory is finite because the number of elliptic CY threefolds is bounded (Kim, Vafa, and Xu, 2024, Tarazi and Vafa, 2021); GG-6 is finite because the OGN-5 arithmetic bounds the Diophantine search space. Understanding whether these two finiteness results are related is an open question.

11.6 The four-dimensional alternative: anomaly-free gauged baryon number by added matter

The comparisons above run against constructions that share the six-dimensional machinery and differ at the axioms. One published alternative differs the other way around: it keeps four dimensions and shares only the question. [Benakli, Goodsell, and Goudeau \(2026\)](#) gauge baryon number directly: their $U(1)_{\bar{B}}$ restricts to ordinary baryon number on the Standard-Model fields (quarks $1/3$, leptons and Higgs neutral — leptophobic by construction), and the anomaly that famously obstructs this gauging is canceled by a *secluded sector* of chiral fermions — vector-like under the Standard-Model group, chiral under $U(1)_{\bar{B}}$, their charges fixed jointly by anomaly cancellation and by the renormalizable interactions needed to give every exotic a mass and a decay channel. The gauge boson is Higgsed by a charge- $3/2$ singlet scalar, $m_{Z'} \simeq \frac{3}{2} g_X v_S$, and viability — exotic decays, no stable colored relics, a dark-matter parity — forces an enlarged scalar sector whose accidental-symmetry breaking yields a light pseudoscalar that carries much of the dark-matter phenomenology.

Two things should be conceded at full strength, because they are true. First, the construction is locally consistent: all six of its $U(1)_{\bar{B}}$ anomaly coefficients vanish exactly on the printed spectrum (their appendix A gives the general solution; the coefficients evaluate to exact zeros on the printed charges). Second, it is *globally* consistent on all closed spin four-manifolds: for a direct-product group $SU(3) \times SU(2) \times U(1)^m$ the spin-bordism classification of [Davighi, Gripaio, and Lohitsiri \(2020\)](#) leaves a single torsion obstruction — Witten’s classical $SU(2)$ parity — and the spectrum carries twenty $SU(2)_L$ -doublet Weyl states, an even number. The four-dimensional route is therefore not merely viable; it is anomaly-free in the same complete sense the present paper claims for BI-6, on its own, four-dimensional, class of backgrounds. The fork between the two constructions is not consistency. It is structure: Table 10 states the full comparison on inputs, and the four paragraphs that follow walk its structural rows.

Table 10: The four-dimensional baryonic completion of [Benakli, Goodsell, and Goudeau \(2026\)](#) against GG-6/BI-6, compared on inputs — the same discipline as Table 9. BGG entries from their Tables I–VI; GG-6 entries from the present paper and its companions ([Arisaka, 2026, A2-GG6](#); [Arisaka, 2026, P1-KK](#); [Arisaka, 2026, C1-Kaxion](#)). The row that carries the argument is the anomaly route: added matter against a topological term, and every other row follows from that one difference.

Feature	BGG (arXiv:2607.12009)	GG-6 / BI-6
Ambient dimension	4	6 (two non-spatial: scale and angle)
Bulk geometry	none	$M_4 \times I_r \times S^1_\theta$, scale-profile metric
Gauge content	$SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_{\bar{B}}$	$\text{Spin}(1,5) \times SU(5) \times U(1)_B \times U(1)_{Q_i}$
Origin of the Abelian factor(s)	postulated: a phenomenological design choice	constitutionally admissible pair (Theorem 6.22); adoption on empirical grounds
Anomaly route	added chiral matter: the secluded sector	topological term: BI-6 inflow, no added charged matter (a gauge-singlet completion sector closes the gravitational books, Section 7)

Feature	BGG (arXiv:2607.12009)	GG-6 / BI-6
Local anomaly	six coefficients vanish exactly on the printed spectrum	$I_8 = \frac{1}{2}X_4^2$, rank-one (Theorem 5.1)
Global anomaly	bordism torsion a single $\mathbb{Z}_2$ (Davighi, Gri-paios, and Lohitsiri, 2020); Witten parity even ($N_2 = 20$)	$\pi_6(\text{SU}(5)) = 0$; Wu–Chern–Simons phase computed on the intended class (Theorem 6.2)
Tensor sector	absent	rank-one unimodular B_2 ; refinement unique (Theorem 6.13)
Continuous dials	seventeen-dimensional scan space; benchmarks	zero; the OGN-5 ledger (1, 3, 5; 7, 35)
Z' mass	Higgsed, $m_{Z'} \simeq \frac{3}{2}g_X v_S$; scanned, 3.85–37 TeV across benchmarks	first Kaluza–Klein note of the Dirichlet-terminated $\text{U}(1)_B$ at $M_{\text{lock}} \approx 3.85$ TeV, from $K = 35$
Baryonic coupling	g_X free or unification-biased (0.027–0.34)	$g_B = 0.565$, derived (Theorem 5.8)
Companions at the Z'	secluded quarks and leptons, extended scalars, a light pseudoscalar — “uncovered in stages”	the radial lock cluster; no charged exotics (the singlet completion of $4\Omega = 100$ tensors and $32\Omega = 800$ Weyl fermions carries no Standard-Model charge)
Dark matter	Majorana secluded neutrino, 0.11–1.2 TeV, parity-protected; direct detection	the tensor remnant (Kaxion): a single line near 1.95 GHz in the window fenced by K ; haloscopes
Proton stability	forbidden before $U(1)_{\bar{B}}$ breaking; post-breaking suppression assumed	exact perturbative selection rule (holonomy-descent step fenced)
Fifth force	none (broken at $v_S \gtrsim 10^4$ GeV)	none (boundary condition; no massless carrier)
Six-dimensional lift	meets a nontrivial π_6 check: $\pi_6(\text{SU}(3)) = \mathbb{Z}_6$, $\pi_6(\text{SU}(2)) = \mathbb{Z}_{12}$ (Mimura and Toda, 1963, Toda, 1962)	clear: $\pi_6(\text{SU}(5)) = 0$ in the stable range

What cancels the anomaly. Their answer is added chiral matter; ours is a topological term with no added *charged* matter — the gauge-singlet completion sector of Section 7, $4\Omega = 100$ tensors and $32\Omega = 800$ Weyl fermions, closes the gravitational books and carries no Standard-Model charge. Their own structural finding sharpens the contrast better than any sentence of ours could: anomaly cancellation alone is cheap — the anomaly equations “admit many solutions” — and it is *viability* that is expensive, dragging in the secluded sector, the enlarged scalar potential, and the light pseudoscalar just to keep the books balanced. BI-6 pays for the same consistency with one tensor already present for other reasons, and adds nothing else.

What is dialed. Their analysis scans a seventeen-dimensional parameter space and reports benchmarks; the baryonic coupling is unification-biased in one variant and a free dial in the other ($g_X \approx 0.34$ at low scale in the first, 0.027–0.117 in the second). Here the Abelian coefficient $-2/5$ is the trace-ratio lock of the rank-one factorization (Theorem 5.1), the coupling $g_B = 0.565$ is the capacity readout of Theorem 5.8, and the scale is the throat depth $K = 35$ of the OGN-5 ledger: zero continuous dials anywhere in the chain.

How many Abelian factors, and why. They postulate one; the constitution admits the three- and four-component bulks of Theorem 6.22, and the four-component version is adopted on the empirical grounds of Section 6.6. Their construction is, in effect, a four-dimensional relative of the one-Abelian reading, with the $SU(5)$ organization and every $U(1)_{Q_i}$ anchor forgone.

What a six-dimensional lift would cost. They have no tensor sector, no rank-one lattice, no Wu structure; embedding their gauge content as the boundary of a bulk in the present paper’s manner would require each of those as an independent axiom (Theorem 6.13 is what spares BI-6 the choice) — and would then meet the obstruction of hypothesis (H2): $\pi_6(SU(3)) = \mathbb{Z}_6$ and $\pi_6(SU(2)) = \mathbb{Z}_{12}$ are non-trivial (Mimura and Toda, 1963, Toda, 1962), where $\pi_6(SU(5)) = 0$ in the stable range. The choice of $SU(5)$ is what makes the global theorem of Section 6 available at all.

The collider face. Their non-unified Benchmark 3 places the Z' at 3.85 TeV — numerically the lock scale M_{lock} to the printed digit — with $g_X \approx 0.027$ and the breaking scale scanned. The coincidence is a scan output: staging, never evidence. At equal mass the coupling-squared ratio is $(0.565/0.0275)^2 \approx 420$, but a coupling ratio is not a rate ratio: on the Dirichlet condition of clause (C9) the baryonic vector has no endpoint overlap with the light quarks and is not produced by them at tree level at all, so the rate comparison the ratio invites does not arise (on the Neumann reading of earlier revisions, with the cohort’s overlap factor and branching fractions, the two dijet rates were comparable and ours in $t\bar{t}$ roughly ninety times theirs, both below current sensitivity). The observable discriminant is the pattern. Their Z' is a narrow, quark-universal line with most of its anomaly-canceling spectrum above it (“uncovered in stages,” in their own phrase) and its dark sector below it; ours is a color octet at the same mass — broad, top-saturated, brighter than their line by several hundred in production and by a factor of order forty in dijets on the collider companion’s own rates — with a graviton and an unproduced baryonic vector degenerate beneath it and no exotics anywhere (the cohort inputs behind these ratios: overlap $\xi_{UV} = -0.194$, $BR(t\bar{t}) = 0.94$; Arisaka, 2026, P1-KK). And the mass is locked against scanned, the coupling derived against dialed.

The dark sector. Their dark matter is the lightest Majorana secluded neutrino, at 0.11–1.2 TeV across their benchmarks, stabilized by a discrete parity, with the relic set by pseudoscalar pairs, co-annihilation, or W^+W^- — direct-detection territory. Ours is the light remnant of the BI-6 tensor: a single line near 1.95 GHz in the window fenced by K (Arisaka, 2026, C1-Kaxion). The two candidates are separated by sixteen to seventeen orders of magnitude in mass, and the search programs do not overlap: a confirmed recoil from a TeV-scale state coupled through a baryonic Z' would be evidence for their class and is something this framework cannot produce; a confirmed line in the fenced window is the converse. Either dark-matter experiment discriminates the two completions outright.

One convergence completes the picture. Their $U(1)_{\tilde{B}}$ is broken at $v_S \gtrsim 10^4$ GeV and supports no long-range force, so the orthodox completion arrives independently at the same composition-independence null this framework obtains from the boundary termination. The null therefore does not discriminate between the completions; the collider pattern and the dark sector do.

12 From AdS/CFT and Randall–Sundrum to OGGEF: shared geometry, transformed physics

The six-dimensional GG-6 bulk with its scale-profile metric, logarithmic radial coordinate, and bulk-to-boundary projection is structurally indebted to two of the most influential ideas in modern string theory: the anti-de Sitter/conformal field theory (AdS/CFT) correspondence (Gubser, Klebanov, and Polyakov, 1998, Maldacena, 1998, Witten, 1998) and the Randall–Sundrum (RS) model of warped extra dimensions (Randall and Sundrum, 1999). GG-6 retains the core mathematics of both but replaces the underlying physics with OGGEF—the Operational/Open Geometry–Gauge Equivalence Principle. This section explains what is borrowed, what is replaced, and why the replacement constitutes a departure rather than a minor variation.

12.1 The AdS/CFT correspondence: a brief review

The AdS/CFT correspondence, proposed by Maldacena in 1997 (Maldacena, 1998) and made precise by Gubser, Klebanov, and Polyakov (Gubser, Klebanov, and Polyakov, 1998) and by Witten (Witten, 1998), establishes an exact duality between:

- (i) a gravitational theory (Type IIB string theory or supergravity) on $\text{AdS}_5 \times S^5$, and
- (ii) a four-dimensional conformal field theory ($\mathcal{N} = 4$ super-Yang–Mills with gauge group $\text{SU}(N_c)$) living on the boundary of AdS.

The radial coordinate z of AdS encodes the energy scale of the boundary CFT: the boundary at $z \rightarrow 0$ corresponds to the UV, and the interior at $z \rightarrow \infty$ corresponds to the IR. Bulk fields act as sources for boundary operators, and the on-shell bulk action generates the boundary generating functional:

$$Z_{\text{CFT}}[\phi_0] = \exp\left(-S_{\text{bulk}}[\phi|_{z \rightarrow 0} = \phi_0]\right). \quad (115)$$

Holographic renormalization (de Haro, Skenderis, and Solodukhin, 2001) makes this correspondence precise by regulating the on-shell action near the boundary and identifying boundary counterterms with renormalization-group counterterms.

Three key features of AdS/CFT are relevant for comparison:

- (a) The radial direction is dual to the RG scale.
- (b) An on-shell bulk functional generates boundary correlators.
- (c) The duality is *exact* (in principle) but requires supersymmetry and a large- N_c limit for computational control.

12.2 The Randall–Sundrum model: a brief review

The Randall–Sundrum model (Randall and Sundrum, 1999) introduces a slice of AdS_5 bounded by two branes:

$$ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad y \in [-\pi R_c, \pi R_c], \quad (116)$$

where k is the AdS curvature scale and R_c is the compactification radius. The UV brane sits at $y = 0$ and the IR brane at $y = \pi R_c$. The exponential warp factor generates a large hierarchy between the Planck scale (set by the UV brane) and the TeV scale (set by the IR brane):

$$\frac{m_{\text{IR}}}{m_{\text{UV}}} = e^{-k\pi R_c}. \quad (117)$$

The Randall–Sundrum model was the first concrete realization of the idea that an extra dimension with warp factor can solve the gauge hierarchy problem without supersymmetry.

Three features are relevant for comparison:

- (a) The fifth dimension is an interval (not a circle), with different physics at the two boundaries.
- (b) The warp factor generates a mass hierarchy geometrically.
- (c) The model does not require supersymmetry (though it is compatible with it).

12.3 What GG-6 borrows: the shared mathematics

GG-6 inherits three mathematical structures from these constructions.

(i) The logarithmic radial coordinate. The GG-6 radial coordinate

$$r = \ln\left(\frac{E_{\text{GUT}}}{\mu}\right) \quad (118)$$

is the direct analogue of the AdS/CFT radial coordinate and the Randall–Sundrum fifth dimension. In all three frameworks, the extra dimension geometrizes the renormalization-group scale. The mathematical content is the same: the RG semigroup $\mathcal{G}_{\lambda_1} \circ \mathcal{G}_{\lambda_2} = \mathcal{G}_{\lambda_1 \lambda_2}$ is additively represented by a logarithmic coordinate.

(ii) The warped metric. The GG-6 metric

$$ds_6^2 = e^{-2A(r)} g_{\mu\nu}(x) dx^\mu dx^\nu + dr^2 + R_\theta^2 d\theta^2 \quad (119)$$

with $A(r) = kr$ is structurally identical to the RS metric (116), with the replacement $|y| \rightarrow r$ and the addition of the compact circle S_θ^1 . The warp factor e^{-2kr} generates the mass hierarchy between the GUT scale and the infrared boundary.

(iii) The bulk-to-boundary functional. In AdS/CFT, the on-shell bulk action generates the boundary generating functional (115). In GG-6, the bulk functional generates the boundary effective action through the OGGE channel:

$$\mathcal{D}_{\text{OGGE}} : \rho_{\text{Bulk}} \longmapsto (\rho_\partial, \Phi_{t,s}, \{\mathcal{I}_\alpha\}). \quad (120)$$

Both are “holographic” in the sense that lower-dimensional boundary physics is encoded in and generated by higher-dimensional bulk physics.

12.4 What GG-6 replaces: the transformed physics

Despite sharing the mathematical infrastructure, GG-6 makes three fundamental conceptual departures from AdS/CFT and Randall–Sundrum.

12.4.1 Departure 1: from exact duality to operational projection

In AdS/CFT, the bulk and boundary theories are *exactly dual*: they are different descriptions of the same physics. The boundary CFT is a complete, self-contained theory; the bulk gravity is another complete, self-contained theory. Neither is “more fundamental” than the other. The duality is *ontologically symmetric*.

In GG-6/GDOS, the relationship is *asymmetric*. The six-dimensional bulk is the constitutionally complete theory. The four-dimensional boundary is an *operational projection* obtained by tracing over the inaccessible bulk degrees of freedom. The boundary theory is derived from the bulk, not dual to it. The mathematical implementation is not a duality but a completely positive trace-preserving (CPTP) channel:

$$\rho_{\partial} = \text{Tr}_{\text{env}}(\rho_{\text{Bulk}}), \quad (121)$$

followed by the exact Nakajima–Zwanzig memory-kernel equation for the reduced boundary dynamics. The boundary observer does not “see” an equivalent dual theory; the boundary observer sees a *reduced* theory with environmental decoherence, memory effects, and measurement-induced state updates.

The difference is fundamental. AdS/CFT is a statement about two complete theories. OGGEF is a statement about how an incomplete observer extracts lawful physics from a more complete bulk.

12.4.2 Departure 2: from conformal symmetry to open-system dynamics

In AdS/CFT, the boundary theory is a conformal field theory with exact conformal symmetry (at least in the large- N_c , strong-coupling limit). The boundary dynamics is unitary, reversible, and exactly solvable in many cases.

In GG-6/GDOS, the boundary dynamics is generically *non-unitary*, *irreversible*, and *open*. The Nakajima–Zwanzig equation has a memory kernel and an inhomogeneous term; the Markovian limit yields the GKLS (Gorini–Kossakowski–Lindblad–Sudarshan) master equation with a dissipator. The mesoscopic coarse-graining yields the GDOS continuity equation with gradient, solenoidal, and diffusive current components.

This replacement is physically motivated: actual scientific observations are performed by finite apparatuses interacting with environmental degrees of freedom. No real experiment is a conformal field theory. The OGGEF channel produces a boundary theory that is *operationally realistic* rather than conformally symmetric.

12.4.3 Departure 3: from string-theoretic completeness to axiomatic measurement theory

In **AdS/CFT**, the bulk-to-boundary map is a tool for computing correlators, partition functions, and entanglement entropies. It does not, by itself, address the measurement problem of quantum mechanics. The question “what is a measurement?” is external to the formalism.

In **GG-6/GDOS**, the bulk-to-boundary map *is* the measurement theory. **AX-3** (OGGEP) constitutionally requires that the theory provide its own operational dictionary. The boundary instruments

$$\mathcal{I}_\alpha(\rho_\partial) = \text{Tr}_A \left[(\mathbf{1} \otimes \Pi_\alpha) U_{SA}(\rho_\partial \otimes \sigma_{\text{app}}) U_{SA}^\dagger (\mathbf{1} \otimes \Pi_\alpha) \right] \quad (122)$$

are not interpretational additions; they are derived from the 6D→4D projection. The Born rule, the instrument formalism, and the Hilbert-space structure of four-dimensional quantum mechanics all emerge as consequences of the operational channel.

No other aspect of **GG-6** departs as sharply from the string-theoretic tradition as this one. In string theory, quantum mechanics is a prerequisite for the theory: one quantizes the worldsheet, computes amplitudes, and takes the classical limit. In **GG-6/GDOS**, quantum mechanics is a *consequence* of the theory: the boundary observer sees quantum behavior because the 6D→4D channel is a CPTP map that produces density operators, instruments, and Born-rule statistics.

12.5 Comparison table: AdS/CFT, Randall–Sundrum, and GG-6/GDOS

Table 11: Three constructions that share a warped extra dimension, set side by side so the shared machinery and the different commitments are both visible. All three use the same mathematics of a warped metric; they differ in dimension, in what fixes the warp factor, and in whether the extra direction is spatial. The third of those is the one to attend to, because it is where this framework departs from both neighbors: the radial direction here carries scale rather than length, and that reading is what makes the warp factor computable rather than chosen.

Feature	AdS/CFT	Randall–Sundrum	GG-6 / GDOS
Spacetime dim.	5 (AdS ₅)	5	6 ($M_4 \times I_r \times S_\theta^1$)
Radial coord. meaning	RG scale / holog. depth	Hierarchy direction	$r = \ln(E_{\text{GUT}}/\mu)$
Warp factor	$e^{-2z/L}$	$e^{-2k y }$	e^{-2kr}
Boundary	Conformal bdry of AdS	UV and IR branes	M_4 at $r = 0$
Bulk–boundary relation	Exact duality	Localization of gravity	OGGEP operational projection
SUSY	Required ($\mathcal{N} = 4$ SYM)	Not required	Not required
Boundary dynamics	Unitary CFT	Standard QFT on brane	Open-system (GKLS/GDOS)
Measurement theory	External to formalism	External to formalism	Derived from 6D→4D channel
Gauge group	$\text{SU}(N_c)$ (any N_c)	Model-dependent	$\text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$
UV completion	String theory at E_{Pl}	Not specified	Constitutional at E_{GUT}
Anomaly cancellation	Matched across the duality	Not addressed	BI-6 (non-SUSY GS mechanism)

Feature	AdS/CFT	Randall–Sundrum	GG-6 / GDOS
Hierarchy solution	AdS/CFT duality	Warp factor	Warp factor + OGN-5
Landscape	Large (many AdS vacua)	No landscape issue	Finite (Diophantine)
Born rule	Assumed	Assumed	Derived

12.6 What string theorists contributed and what **GG-6** adds

The intellectual debt to the string theory community is substantial and should be stated without ambiguity.

What **GG-6** inherits from string theory:

- (i) The Green–Schwarz mechanism for anomaly cancellation (Green and Schwarz, 1984).
- (ii) The idea that an extra radial dimension can geometrize the renormalization-group scale (holographic RG, Maldacena–Gubser–Witten, 1997–1998).
- (iii) The warped metric as a hierarchy-generating mechanism (Randall and Sundrum, 1999).
- (iv) The technology of anomaly polynomials, descent equations, and charge lattices developed over four decades of string/F-theory research.

Without these contributions, **GG-6** could not exist in its present form.

What **GG-6** adds beyond string theory:

- (i) A constitutional axiom system (AX-1–AX-4, POS-1–POS-7) that derives the bulk geometry, gauge group, and anomaly structure from admissibility requirements alone, without postulating strings, branes, or compactification manifolds. The fundamental excitations remain point-like local field quanta (AX-1 locality and causality), not extended objects; because the two further dimensions are non-spatial — a scale dimension and a gauge dimension — **GG-6** remains a point-particle framework even as it geometrizes the gauge sector.
- (ii) The **OGGEP** channel: a lawful operational map from bulk to boundary that turns quantum measurement from an external postulate into a derived consequence of the 6D→4D projection.
- (iii) The **OGN-5** arithmetic: a discrete integer ledger that freezes all dimensionless normalization freedom, replacing the continuous moduli spaces of string compactifications with a finite discrete structure.
- (iv) The **BI-6** identity: a non-supersymmetric six-dimensional Green–Schwarz mechanism with a constitutionally determined coefficient vector, achieving anomaly cancellation without SUSY, strings, or a Planck-scale UV completion.

- (v) The GDOS boundary dynamics: an open-system evolution theory (Nakajima–Zwanzig $\rightarrow$ GKLS $\rightarrow$ GDOS) that unifies quantum mechanics, measurement, and the Born rule as boundary projections of one bulk variational engine.

12.7 The philosophical shift: from ontology to operational law

The most far-reaching difference between GG-6/GDOS and the AdS/CFT tradition is not mathematical but philosophical.

In AdS/CFT, the bulk and boundary are two *equally real* descriptions of the same ontological reality. The holographic principle is a statement about the nature of spacetime itself: the number of degrees of freedom scales with the boundary area, not the bulk volume.

In GG-6/GDOS, the bulk is the *constitutionally complete* description. The boundary is what a *finite observer* can register. The distinction between bulk and boundary is not ontological but *operational*: it is the distinction between the full lawful dynamics and the reduced dynamics accessible to an agent who cannot retain all degrees of freedom.

This shift has a concrete consequence for the measurement problem. In AdS/CFT, the measurement problem is no easier than in ordinary quantum mechanics, because the boundary theory is itself a unitary quantum theory. In GDOS, the measurement problem is dissolved: boundary observations are CPTP channels with instruments, and the Born rule is a consequence of the channel structure, not an additional postulate.

Whether this philosophical shift is ultimately correct is an empirical question. But the conceptual clarity of this shift is testable: if quantum mechanics is the operational shadow of a higher-dimensional constitutional law, then the apparent “strangeness” of quantum measurement is not strange at all—it is the inevitable consequence of projecting a closed bulk onto an open boundary.

12.8 AdS/CFT, Randall–Sundrum, and BI-6 as Stratum 2 specializations of the GGEP-closed axiom layer

The comparisons developed in Sections 10–12 can be sharpened by adopting the constitutional-classification vocabulary established in GG-A5-GDOS (GG-A5-GDOS Part VIII, Figure 5; Arisaka, 2026, A5-GDOS). We do so explicitly here, because the classification places the present anomaly theorem in a sharply defined axiomatic position relative to the broader twentieth-century high-energy edifice, and the position matters both for the scope of what the theorem accomplishes and for the scope of what remains to be done by the companion papers.

The GGEP-closed axiom layer (Stratum 2). By the constitutional classification of GG-A5-GDOS Part VIII, the *GGEP-closed axiom layer* is the subset $\{\text{AX-1}, \text{AX-2}\}$ of the four-axiom GG-Constitution. It activates local causality (AX-1, LC) and the geometry–gauge equivalence principle (AX-2, GGEP) but neither the operational/open extension (AX-3, OGGE) nor the maximum-symmetry / minimum-freedom selection (AX-4, MSMF). Stratum 2 of the constitutional tower is the framework class admitted by exactly this axiom subset.

Frameworks placed on Stratum 2. The frameworks that sit on Stratum 2 of the constitutional tower include, in chronological order:

- the Bianchi identity in its classical form (Bianchi, 1902);
- general relativity (Einstein, 1915), whose composite geometry–gauge content is the Einstein tensor itself $G_{\mu\nu}$ with the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ as the local conservation law;
- Yang–Mills gauge theory (Yang and Mills, 1954), which is the prototype geometrization of internal symmetry;
- the Green–Schwarz anomaly-cancellation mechanism (Green and Schwarz, 1984), which is the moment higher-dimensional gauge theory recognized that gauge curvature F and gravitational curvature R must lie inside a single closed four-form on the bulk manifold (Section 10.7 of the present paper);
- superstring theory at the GGEP-closed level (perturbative Type I, IIA, IIB, heterotic; Polchinski, 1998,b);
- the AdS/CFT correspondence (Gubser, Klebanov, and Polyakov, 1998, Maldacena, 1998, Witten, 1998) and its Gubser–Klebanov–Polyakov–Witten dictionary (Section 12 above);
- the Randall–Sundrum warped construction (Randall and Sundrum, 1999) and the family of holographic dualities that descend from it;
- the present BI-6 anomaly theorem, which is the six-dimensional four-component generalization of the Green–Schwarz factorization, and the entire descent + global-completion proof chain of Sections 5 through 6.

All eight items in this list are theorems of the same axiom subset $\{\text{AX-1}, \text{AX-2}\}$; they differ from each other only in which specific bulk geometry they take as input, not in which constitutional clauses they invoke.

What Stratum 2 can derive. The GGEP-closed axiom layer suffices to derive:

1. the existence of a composite geometry–gauge connection on a principal G_{tot} bundle over an admissible bulk manifold (AX-2);
2. the closed gauge-invariant four-form $X_4 = \text{tr}(R \wedge R) - \text{Tr}(F \wedge F) - \dots$ that lives on this composite bundle (Chern–Weil theory applied to the composite connection);
3. the modified Bianchi identity $dH_3 = X_4$ for the tensor sector, in any number of even spacetime dimensions where the anomaly polynomial admits a Green–Schwarz factorization $I_{2n+2} = \frac{1}{2}X_4 \wedge X_{2n-2}$ or, in six dimensions with rank-one factorization, $I_8 = \frac{1}{2}X_4 \wedge X_4$ (Theorem 5.5 of the present paper);
4. one-loop and global anomaly cancellation by inflow, given the Wu–Chern–Simons / Hopkins–Singer / Monnier–Moore differential-cohomology framework on the chosen bulk class (Theorem 6.2 of the present paper);

5. the existence of a bulk-to-boundary projection, in the closed-system limit, on any bulk geometry with a conformal boundary (AdS/CFT as the canonical example).

This is the theorem class admitted by Stratum 2. It is large enough to contain the classical-topological backbone of the GS mechanism, of AdS/CFT, of holographic dualities, of Randall–Sundrum, and of the present BI-6 construction (the quantum-dynamical sectors that run on top of that backbone are a separate matter, scoped below).

What Stratum 2 cannot derive without AX-3. The following structural facts are *not* theorems of the GGEP-closed axiom layer alone; they require activation of AX-3 (OGGEP):

1. an operational dictionary $\mathcal{D}_{\text{OGGEP}} : (\mathcal{H}_{\text{Bulk}}, \rho_{\text{Bulk}}, U_{\text{Bulk}}) \mapsto (\mathcal{H}_{\text{Bdy}}, \rho_{\text{Bdy}}, \Phi_{t,s}, \{\mathcal{I}_\alpha\})$ from inaccessible bulk data to registered boundary observables, with the bulk-to-boundary projection a completely positive trace-preserving (CPTP) channel rather than a unitary equivalence;
2. the seven textbook postulates of quantum mechanics — Hilbert space, the Born rule, the Lüders projection, decoherence, einselection, the violation of Bell inequalities — recovered as theorems of the constitutional projection rather than as separate postulates (GG-A5-GDOS Part III (Arisaka, 2026, A5-GDOS));
3. the second law of thermodynamics as the data-processing inequality of the boundary CPTP channel, with the Spohn quantum H -theorem as the operator-level realization;
4. the dissolution of the black-hole information paradox by identifying Hawking radiation as the output of the OGGEP channel at the event horizon, with the Page curve as a bipartite-pure-state bookkeeping argument and the 2020 islands formula as the bulk-side geometric reconstruction.

The absence of AX-3 from Stratum 2 is the constitutional source of the QM-foundations gap that has divided the string-theory and QM-foundations communities for fifty years.

What Stratum 2 cannot derive without AX-4. The following structural facts are not theorems of GGEP-closed alone; they require activation of AX-4 (MSMF):

1. a unique vacuum: GGEP-closed admits any bulk geometry whose Bianchi identity factorizes through a single tensor sector, including the perturbative string landscape of $\sim 10^{500}$ inequivalent compactifications (Douglas, 2003, Susskind, 2003). AX-4 selects the unique bulk that minimizes continuous parameter freedom subject to the maximum-symmetry constraint, eliminating the landscape by construction. In the present paper, the six-dimensional bulk $\mathcal{M}_6 = \mathcal{M}_4 \times I_r \times S_\theta^1$ of Arisaka (2026, A2-GG6) is the AX-4-selected representative of GGEP-closed-admissible bulks;
2. the OGN-5 integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ as the unique Diophantine solution of the global consistency of BI-6 (Section 9 of the present paper). Without AX-4, the integers would be a chosen ledger; *with* AX-4, they are the unique ledger compatible with the rest of the constitutional clauses.

On this classification, what the cohort program adds at Stratum 2 is **AX-4**, and it is **AX-4** that removes the continuous freedom a landscape of vacua represents. That is a statement about what the two frameworks assume, not a claim that one has failed: a framework without **AX-4** is not required to have a unique vacuum, and the string program has never claimed that it must.

The Schrödinger–GKLS analogy at the holographic layer. The relationship between AdS/CFT (a GGEP-closed framework on Stratum 2) and the cohort 6D-to-4D projection (a Full-Constitution framework on Stratum 4 that uses all four axioms) is precisely the holographic-layer analogue of the relationship between the Schrödinger equation (closed-system dynamics) and the Gorini–Kossakowski–Lindblad–Sudarshan master equation (open-system dynamics). Stated as a parallel pair:

$$\begin{array}{ccc} \text{GKLS (open)} & \xrightarrow{\gamma_\alpha \rightarrow 0} & \text{Schrödinger (closed)}, \\ \text{Cohort 6D–4D projection (open)} & \xrightarrow{\text{AX-3 off}} & \text{AdS/CFT duality (closed)}. \end{array}$$

The structural content of both limits is the same: switching off the open-system axiom (the GKLS dissipators γ_α at the dynamical layer, the operational coarse-graining **AX-3** at the holographic layer) collapses an open CPTP evolution to a closed unitary evolution. AdS/CFT is the closed-bulk limit of the cohort 6D-to-4D projection; this is why AdS/CFT realizes holography as information-preserving *exact duality* (the boundary CFT and the bulk gravitational theory share the same Hilbert space, with the on-shell bulk action generating boundary correlators exactly), whereas the cohort projection realizes holography as information-coarsening CPTP map (the boundary state is mixed even when the bulk state is pure, with the Born rule a Gleason-type theorem on the boundary algebra and the second law a Lindblad-type monotonicity statement on the boundary entropy). Both readings are correct on their respective axiom subsets; neither is the other’s competitor. The present paper’s anomaly theorem closes the GGEP-closed prerequisite for either reading. GG-A5-GDOS §69.4 (Arisaka, 2026, A5-GDOS), where the second law appears as a data-processing inequality, develops the analogy in full.

Where this places BI-6. The constitutional classification therefore places the central theorem of the present paper precisely: **BI-6** anomaly freedom is a Stratum 2 theorem of the constitutional tower. It is the strongest Stratum 2 result the framework needs. It does *not* yet establish the operational OGGEF channel that turns the bulk theory into a boundary observational program — that is the job of the companion paper GG-A5-GDOS (Arisaka, 2026, A5-GDOS). It does *not* yet select the unique GG-6 bulk uniquely among GGEP-closed-admissible bulks — that is the job of **AX-4** as deployed in GG-A2-GG6 (Arisaka, 2026, A2-GG6) the dimensional-minimality theorem (dimensional minimality), Theorem on metric uniqueness, and the **OGN-5** closure of the present paper’s Section 9. But it does, at this stratum, deliver the global anomaly consistency that is the gating prerequisite for both the **AX-3** and the **AX-4** applications. In the language of the constitutional flowchart of GG-A5-GDOS Figure 5: **BI-6** closes the lock on the bottom-right cell of Stratum 2.

Historical analogy: Riemann to Einstein, Green–Schwarz to the cohort. The reader who finds this constitutional reading unfamiliar may find the following historical parallel useful. Riemannian geometry was developed by [Riemann \(1854\)](#) as pure mathematics, sixty years before Einstein ([Einstein, 1915](#)) identified the physical setting (gravitation as curvature of spacetime) in which Riemannian geometry became operative. The mathematical content of Riemann’s *Habilitationsschrift* was correct, complete, and unchanged by Einstein’s later constitutional identification; what changed was the *recognition* of which physical question the mathematics was the right tool for. The relationship between Green–Schwarz / AdS/CFT / Randall–Sundrum and the cohort constitutional setting is structurally analogous, with the temporal direction reversed: the geometric machinery was developed inside string theory between 1984 and 2020, and the constitutional setting in which that machinery is recognized as the GGEP-closed content of the GG-Constitution is identified by the cohort program of 2026. None of the string-theoretic mathematical content is altered or contradicted; what is added is the constitutional setting in which it is recognized as the geometrization of gauge in higher dimensions. The 1984 GS mechanism becomes, in this reading, the moment string theory glimpsed AX-2 (GGEP) at the technical level; the 1997 AdS/CFT correspondence becomes the moment string theory glimpsed the GGEP-closed bulk-to-boundary projection in the closed-system limit; the 2020 islands formula becomes the moment string theory glimpsed the bulk-side geometric reconstruction of what the operational AX-3 channel does on the boundary side. Each move was correct as far as it went, and each was limited only by the axiomatic clauses (AX-3 and AX-4) that the cohort framework now makes explicit.

After all, the test of a constitutional framework is not whether it makes its predecessors wrong but whether it makes their contributions intelligible inside a wider picture. By that test, the constitutional classification places the entire string-theoretic edifice cleanly into the GGEP-closed layer of a constitutional tower whose three remaining layers (AX-1 only at Stratum 1, plus AX-3 at Stratum 3, plus AX-4 at Stratum 4) are the contributions of the cohort program.

12.9 Scope and limits: what the cohort framework does and does not claim about string theory

A senior reader trained on string theory will reasonably ask whether the cohort framework’s claim that “string theory is the geometrization of gauge, not the quantization of geometry” is the right summary of four decades of string-theoretic results. We owe that reader a precise answer.

What the cohort claim DOES include. The GGEP-closed sector of string theory — the descent formalism of [Wess and Zumino \(1971\)](#) and [Becchi et al. \(1976\)](#), the modified Bianchi identity of [Green and Schwarz \(1984\)](#), the characteristic-class content of anomaly cancellation, the differential-cohomology classification of [Hopkins and Singer \(2005\)](#) and [Monnier and Moore \(2019\)](#), and the AdS bulk-to-boundary projection *in the closed-system limit* of the Gubser–Klebanov–Polyakov–Witten dictionary — is captured by AX-2 (GGEP) of the GG-Constitution. This sector is classical-topological in content: the mathematical statements ($dH_3 = X_4$, Stora–Zumino descent, Hopkins–Singer differential cohomology) make no reference to quantum loops in their formulation. The cohort framework recognizes this sector as the AX-2 content of any candidate theory and

elevates the recognition to a constitutional axiom.

What the cohort claim DOES NOT include. The quantum-mechanical content of string theory is a separate body of work that the cohort framework does not subsume. The closed-string vertex operator that yields the graviton mode is a quantum-mechanical operator on the worldsheet conformal field theory; the α' and $1/N$ expansions in AdS/CFT are quantum corrections to the bulk supergravity description; the islands formula of Almheiri–Penington–Stanford–Yang and Penington–Shenker–Stanford–Yang of 2020 (Almheiri *et al.*, 2020, Penington *et al.*, 2020) computing the Page curve of Hawking radiation is a quantum-gravitational result with no classical-topological analogue. These results are not in the GGEP-closed sector; they belong to the quantum-dynamical sector that runs on top of the GGEP-closed backbone. The cohort framework neither replaces nor reproduces this content in the present paper.

Precise scope of the geometrization-of-gauge claim. When we say in this paper that *string theory is the geometrization of gauge in higher dimensions, not the quantization of geometry*, the precise claim is this: the structural backbone that the Green–Schwarz mechanism, AdS/CFT, holographic dualities, and the modified Bianchi identity share — the classical-topological content of differential characters on a higher-dimensional bulk manifold — is the geometrization-of-gauge content forced by AX-2. The string-theoretic quantum dynamics on top of this backbone is a separate sector that the cohort framework neither replaces nor reproduces.

Precise scope of the AdS/CFT analogy. The AdS/CFT-as-AX-3-switched-off-limit framing of Section 12 (and developed in detail in GG-A5-GDOS (Arisaka, 2026, A5-GDOS)) should be read in the same scope-restricted sense: it identifies the structural backbone shared by AdS/CFT and the cohort 6D-to-4D projection at the GGEP-closed layer. It does *not* claim a precise GKP/Witten-level dictionary (Gubser, Klebanov, and Polyakov, 1998, Witten, 1998) from the 6D GG-6 bulk action to the IIB SUGRA action on $\text{AdS}_5 \times S^5$ in the AX-3-off limit; that explicit dictionary, including the matching of the holographic correlator generating functional, of the a - and c -coefficients between bulk SUGRA and boundary CFT, of Wilson loops via minimal surfaces, and of the Ryu–Takayanagi entanglement entropy formula, is recorded as Open Problem O12 of Section 17.

Why we make this qualification explicit. The cohort framework’s contribution at the GGEP-closed layer is intellectually serious: it elevates the geometric content of forty years of string-theoretic results to an axiomatic recognition. That contribution does not require, and does not benefit from, the suggestion that the quantum content of string theory has been displaced or subsumed. The honest summary is: the cohort framework adds a constitutional setting in which the classical-topological backbone is recognized as AX-2; the quantum sector of string theory remains the body of work that the string community has built over four decades, and the cohort framework neither replaces nor disputes it.

13 Swampland compatibility of GG-6 and BI-6

The swampland program of Vafa (2005), Brennan, Carta, and Vafa (2017), and Tarazi and Vafa (2021) proposes that not every internally consistent effective field theory of gravity is the low-energy limit of a UV-complete quantum theory of gravity: those that are not lie in the “swampland”. The program has proposed a family of conjectures (no global symmetries, completeness hypothesis, weak-gravity conjecture, distance conjecture, de Sitter conjecture, cobordism conjecture) that any candidate quantum-gravity EFT must satisfy. The conjectures were derived from string-theoretic intuition, and their universality outside string theory is itself a matter of active debate.

A reader trained on the swampland program will reasonably ask whether the non-stringy cohort framework GG-6 + BI-6 satisfies these constraints. We check the framework against the five most widely accepted conjectures and state the result honestly.

Conjecture 1: No global symmetries. Every continuous global symmetry of a consistent quantum-gravity EFT must be either gauged or broken at high energy. In the cohort framework, the bulk gauge group is $G_{\text{Bulk}} = \text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Qi}$ with all four factors gauged by AX-2 (GGEP), and the constitutional admissibility theorem (Theorem 6.22) confirms that the boundary $\text{U}(1)_B$ baryon-number symmetry and the optional $\text{U}(1)_{Qi}$ information-current symmetry both descend from bulk gauge structure. The cohort framework therefore has no ungauged continuous *structural* symmetries. The gravitational-completion sector, however, carries two classes of accidental continuous global symmetry that this conjecture is precisely about: the flavor symmetry of its gauge-singlet fermions (as any large singlet sector does), and the higher-form symmetries of its free spectator tensors (the currents $H_3^{(i)}$ and their duals). Their fate — lifted by bulk masses, gauged by couplings the minimal action omits, or broken by the boundary conditions of the intended class — is completion-sector physics that is not settled here. Grading this entry honestly: **structurally compatible for the gauge sector; resolution deferred for the completion sector** (Objection O-BI6-59).

Conjecture 2: Completeness hypothesis. Every charge in the gauge lattice must be populated by some physical state. The cohort framework realizes this in the integer OGN-5 ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$, which fixes the bulk charge spectrum to integer multiples of the primitive lattice with $J = 7$ giving the symmetric closure cardinality $\{0, \pm 1, \pm 2, \pm 3\}$. Whether every lattice site in this range is populated by a physical state is the subject of the benchmark spectrum construction of Section 7 and Theorem 7.9; the spectrum is constructed explicitly and the lattice is saturated by the cohort fermion reservoir for every $\Omega \in 25\mathbb{Z}$. **Satisfied at the existence level.**

Conjecture 3: Weak-gravity conjecture (WGC). For every $\text{U}(1)$ gauge symmetry in a quantum-gravity EFT, there must exist a particle with charge q and mass m satisfying $gqM_{\text{Pl}}/m \geq 1$ (in appropriate units), so that extremal black holes can decay. For the cohort bulk, the WGC constraint applies to $\text{U}(1)_B$ and $\text{U}(1)_{Qi}$. The cohort spectrum delivers particles with charges $\pm 1, \pm 2, \pm 3$ under $\text{U}(1)_B$ and similarly under $\text{U}(1)_{Qi}$; whether the lightest charged state in the

descended 4D spectrum saturates the WGC inequality is a calculation that requires the boundary spectrum derivation of GG-A5-GDOS, GG-A7-Qi4, and GG-A9-SM. At the bulk level, the existence of charged states populating the lattice is established; the precise mass-to-charge ratio remains to be computed. **Open: structurally compatible, quantitative check deferred** to the boundary spectrum derivation.

Conjecture 4: Distance conjecture (swampland distance). Every infinite-distance limit in moduli space is accompanied by a tower of light states. The cohort framework has *no infinite-distance moduli limits*: the scale interval is $r \in [0, K]$ with $K = 35$ frozen by the OGN-5 ledger, the compact direction is $\theta \in [0, 2\pi)$ with no continuous radius parameter, and AX-4 eliminates the moduli space of bulk geometries to a single point (the cohort canonical bulk). There is no infinite-distance limit at which the conjecture could be tested. **Vacuously satisfied** (no moduli space to traverse).

Conjecture 5: Cobordism conjecture. In a consistent quantum-gravity EFT, the bordism classes of the theory’s backgrounds must be trivializable in the full theory: any non-trivial class defines a conserved global charge, which must be broken by defects or vanish (Brennan, Carta, and Vafa, 2017). For the cohort GG-6 bulk with structure group $\text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$, the relevant bordism groups in low dimensions are standard (Freed and Hopkins, 2021): $\Omega_n^{\text{Spin}}(\text{pt}) = \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0$ for $n = 0, \dots, 7$; gauge-twisted bordism with the cohort G_{Bulk} adds free summands in even degrees — the perturbative anomaly lattice, canceled in Part II — and, in the anomaly degree itself, no torsion beyond what Lemma 6.6 computes to vanish (see also Hamada and Loges, 2024). The structural verification of hypothesis H3 in Theorem 6.2 confirms bordism-trivializability of the anomaly cocycle on the cohort background class. **Satisfied at the structural level**; the exhaustive enumeration of all twisted bordism contributions for the four-component G_{Bulk} is Open Problem O10.

Summary of the swampland audit. Of the five widely accepted swampland conjectures, the cohort GG-6 + BI-6 framework satisfies three (completeness, distance, cobordism) at the structural level; satisfies no-global-symmetries for the gauge sector, with the completion sector’s accidental symmetries recorded as an open exposure (Objection O-BI6-59); and the fifth (weak-gravity) is structurally compatible with the precise quantitative check deferred to the boundary-spectrum derivation in GG-A5-GDOS, GG-A7-Qi4, and GG-A9-SM. The framework is therefore swampland-compatible by the most widely accepted criteria. We omit the de Sitter conjecture from the audit because its status as a universal swampland constraint is itself disputed in the recent literature (Kim, Vafa, and Xu, 2024, Tarazi and Vafa, 2021) and the cohort cosmological inventory $\Omega_\Lambda = 1 - 1/\pi$ of GG-A10-Cosmos (Arisaka, 2026, A10-Cosmos) is the subject of a dedicated treatment there.

After all, the test of a constitutional framework against the swampland constraints is not whether the framework reproduces the specific string-theoretic derivations of those constraints but whether it satisfies the universal content the constraints encode. By that test, the cohort framework is consistent with the universal content the constraints encode — which is not the same as being realized in string theory, a separate and open question — with the caveat that quantitative confirmation of WGC and the explicit cobordism enumeration remain follow-up calculations.

The comparisons developed in Sections 10–12 can be summarized in one sentence: GG-6 inherits the mathematical infrastructure of the Green–Schwarz mechanism, six-dimensional anomaly-polynomial technology, holographic radial coordinates, and warped extra dimensions, while replacing supersymmetry, fundamental strings, Planck-scale completion, exact duality, and the landscape of string vacua with the GG-Theory Constitution, the OGN-5 arithmetic, and the OGGEF operational projection.

What the comparison found. Five differences carry the weight, and they can be stated in a sentence each.

Supersymmetry: the neighboring six-dimensional constructions require it, and this one does not. Almost every other entry in the comparison follows from that one difference, which is why the tables are built on inputs rather than on results.

Fundamental strings: the ten-dimensional mechanism is a statement about a theory of extended objects, and the six-dimensional one is a statement about a field theory on a bulk. The cancellation calculus is identical; what it is a calculus *of* is not.

Completion: the string program takes the Planck scale as the place where the description is completed, and the present framework does not offer a completion at all. That is a limitation and it is recorded as an open problem, not argued away.

Duality: AdS/CFT reads the radial direction as an exact correspondence between a gravity theory in the bulk and a field theory on the boundary. Here the radial direction is read as a literal, irreversible interval of scale. These are different claims about the same geometry, and only one of them can be right about any given system.

Freedom: a construction with a landscape of vacua can absorb a failed measurement by moving within the landscape; a construction with no continuous parameters cannot. This is a difference in how the two frameworks can be tested, and it cuts both ways — the rigid framework is easier to kill, which is an advantage only if it survives.

None of this establishes that one program is better than the other, and the paper does not claim it does. What the comparison establishes is narrower and more useful: the six-dimensional construction is not a rival derivation of the same thing, and it is not a special case of the ten-dimensional one either. It borrows a technique and applies it under a different set of assumptions, and the honest way to evaluate it is on whether those assumptions produce numbers that turn out to be right. Part V takes up that question.

Part V

Discussions

What the anomaly theorem signifies, what interlocks to make it unique, and what is new here; the predictions that can falsify it; the map of what this paper gives the cohort and takes from it; and the honest triage of what remains open

The proofs are now complete. Parts II and III established local and global anomaly cancellation and derived the OGN-5 integers from BI-6; Part IV placed these results against the major frameworks of modern high-energy theory. This Part steps back to ask the broader questions in the canonical cohort order: what the theorem signifies, what interlocks to make it unique and what is new here, where it can be killed, what the paper gives the rest of the cohort and takes from it, and what remains open. The Conclusions follow in Part VI, written in plain language.

A discussion section is where a paper is most tempted to overstate, so it is worth fixing the standard in advance. Three gradings are used throughout and they are used strictly. A result is *derived* if there is a chain of theorems from the constitutional clauses to the statement, with every link written down. It is *conditional* if such a chain exists but rests on a hypothesis that has not itself been established — in which case the hypothesis is named in the same sentence. It is *open* if no theorem in this paper or in any current companion carries the answer, and saying so is not a failure of nerve but the only way the grading means anything.

Section 14 argues the significance and states what interlocks with what, then lists the ten things that are structurally new. Section 15 gives the falsifiers: the specific measurements that would end the framework, with the precision each would need. Section 16 makes the cohort dependencies explicit in both directions. Section 17 triages seventeen open problems into resolved, partially resolved, and genuinely open, and names the seven that are genuinely open without softening any of them.

The last of those is the one a skeptical reader should turn to first.

14 Significance, uniqueness, and what is new

Four things make the anomaly theorem of this paper significant beyond its technical statement.

First, it is the cohort's license to exist. Every downstream paper of the GG-Theory program — the boundary law of GG-A5-GDOS, the pillars of GG-A6-GRIP and GG-A7-Qi4, the Standard-Model codebook of GG-A9-SM, the cosmological inventory of GG-A10-Cosmos, the capstone of GG-A11-Origin — builds on the GG-6 bulk. A six-dimensional chiral theory that failed the anomaly test would not be inaccurate; it would be inconsistent, and everything built on it would be void. The theorem proved here is therefore not one result among many: it is the gate through which the rest of the program walks. That is also why the paper takes the global half of the theorem so seriously:

a bulk theory that is only infinitesimally consistent could still hand the OGGEF boundary channel a topological phase ambiguity, and the lawfulness of the entire boundary program rests on its absence.

Second, it converts arithmetic into physics. The same factorization that closes the anomaly forces the trace-ratio lock $|a/c| = |a/d| = 5/2$ and, through it, the integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$. Those five integers are not bookkeeping: they reappear as $\alpha^{-1}(0) = 2^J + N^2 = 137$ at leading order, as the throat depth that fences the kaxion's frequency window, and as the throat depth that fixes $M_{\text{lock}} \approx 3.85$ TeV. The anomaly test — the same test that selected $\text{SO}(32)$ and $E_8 \times E_8$ in ten dimensions — here selects numbers that experiments can check.

Third, it demonstrates that the Green–Schwarz mechanism does not need supersymmetry. Forty years of six-dimensional anomaly freedom have been written almost entirely inside $\mathcal{N} = (1, 0)$ supergravity. The present paper closes local and global anomaly freedom for a non-supersymmetric chiral theory, with the constitutional clauses doing the selection work that supersymmetry would have done — a proof of concept that the anomaly-inflow tradition extends beyond the supersymmetric island where it was born.

Fourth, it explains why the ledger cannot run. The couplings of physics run with scale; the anomaly coefficients do not, and the cohort's recent campaign (GG-W1-Symmetry; Arizaka, 2026, P5-CPT) has made the reason exact: *holonomy requires closed loops, and an interval has none.* The two non-spatial fibers of the GG-6 bulk are of opposite topological kind. The compact circle S_θ^1 carries all of the bulk's holonomy — quantized, gauge-invariant, protected; the radial interval I_r carries none — only the one-way RG semigroup of POS-5. Anomaly inflow is circle-side bookkeeping: every coefficient matched in this paper is topological pairing data on the fiber that remembers, which is why the ledger's entries are integers, exact at every scale, rather than couplings with anomalous dimensions — the scale-dependence of physics is the *other* fiber's business, and the interval has no holonomy to renormalize. That is why one and the same ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$, fixed once by the factorization at the top of the descent, can surface unchanged at $\alpha^{-1}(0) = 2^J + N^2$, in the throat depth K that fences the kaxion's window, and at the TeV lock — across the whole of cosmic history, the e^{140} of boundary time chronicled in the capstone (Arizaka, 2026, A11-Origin) — with no radiative erosion in between. Exactness here is not an approximation that improves; it is the topology of the fiber on which the ledger is written. The anomaly theorem is thereby the constitution's guarantee that its arithmetic is permanent: *the circle remembers, and the ledger is written on the circle.*

14.1 Uniqueness: what interlocks with what

Uniqueness. The construction is rigid at every joint where a conventional model would have a dial. The coefficient vector $(1, -1, -2/5, -2/5)$ is the unique rank-one factorization of the microscopically computed anomaly matrix (Theorem 5.1), unique up to the overall sign that the inflow convention fixes. The Abelian fraction $2/5$ is the trace ratio of the Standard-Model embedding and is invariant on the one-point moduli space of normalizations that preserve the integer charges (Section 9). The integer family $(5k, 3k)$ collapses to $(5, 3)$ under the MSMF minimality clause (Theorem 9.5); the closure pair (J, K) is then overdetermined — two independent identities, $K = LJ$ and $K = M^2 + N^2 + L^2$, meet at the same integer 35. And the gravitational completion is a 2×2

linear system with the unique solution $n_{\text{SD}} = 4\Omega$, $n_F = 32\Omega$ (exactly eight singlets per tensor). Nothing in this chain can be adjusted without breaking something else in it: that interlocking is the uniqueness claim, stated at the calibrated register of Section 19.3.

14.2 What is new: the ten structural innovations

The contribution of GG-A3-BI6 is to prove that the GG-6 stage defines a consistent quantum theory: a complete local-and-global anomaly theorem for BI-6, together with the integer ledger that keeps the sector finite. We collect here the ten structural innovations that proof delivers, each carrying its cohort-global identifier (§A).

Innovation 1 (I-BI6-1): A complete local-and-global anomaly theorem in six dimensions.

GG-A3-BI6 proves anomaly freedom for BI-6 on the physical intended GG-6 class — not a symbolic restatement of Green–Schwarz but a concrete 6D result (T-BI6-6.3 local, T-BI6-7.2 global) bringing the explicit factorization, the descent proof, the differential-cohomology completion, and the automatic-admissibility theorem under one roof.

Innovation 2 (I-BI6-2): The explicit BI-6 four-form and its single anomaly polynomial.

The four-form $X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - \frac{2}{5}F_B^2 - \frac{2}{5}F_{Q_i}^2$ with $I_8 = \frac{1}{2}X_4^2$ is written explicitly, so the entire anomaly is carried by one composite invariant rather than an open-ended set of counterterms.

Innovation 3 (I-BI6-3): Single-tensor Green–Schwarz cancellation. One two-form tensor B_2 cancels all gauge, gravitational, and mixed anomalies through a rank-one inflow (T-BI6-3.2, T-BI6-6.1), in contrast to the multiple tensor fields of generic 6D supergravity.

Innovation 4 (I-BI6-4): Global anomaly freedom by differential cohomology. The Wu–Chern–Simons / Hopkins–Singer completion of the tensor counterterm establishes global (not merely local) consistency on the intended class (T-BI6-7.2), closing the topological phase ambiguity a purely local proof would leave open.

Innovation 5 (I-BI6-5): The trace-ratio lock $|a/c| = |a/d| = 5/2$. The BI-6 coefficient vector $(1, -1, -\frac{2}{5}, -\frac{2}{5})$ is forced, not fitted: the Abelian trace ratio is determined by the OGN integers (T-BI6-9.1), fixing the anomaly polynomial with no free coefficient.

Innovation 6 (I-BI6-6): The OGN-5 ledger as a theorem. The integer packet $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ is stated and used at theorem level (T-BI6-10.2), with the Diophantine minimality of $(L, N) = (5, 3)$ proved (T-BI6-10.3) — the arithmetic lock that keeps the BI-6 sector finite and sharply constrained, rather than an external input.

Innovation 7 (I-BI6-7): Anomaly freedom as the prerequisite for a lawful boundary law. Global consistency is shown to be a precondition for the OGGE/GDOS bulk-to-boundary map: a globally multi-valued parent would hand the boundary functional an unphysical topological

phase, so the theorem opens the door to the entire boundary program rather than merely repairing a bulk gauge theory.

Innovation 8 (I-BI6-8): Constitutional admissibility of both bulks. GG-A3-BI6 proves that the constitution admits both the three-component $\text{Spin}(1,5) \times \text{SU}(5) \times \text{U}(1)_B$ and the four-component bulk with $\text{U}(1)_{Q_i}$ (T-BI6-7.5): the informational sector is an admissible extension adopted on empirical grounds, not an axiomatic forcing — an honesty distinction the cohort carries downstream.

Innovation 9 (I-BI6-9): Anomaly cancellation as geometric smoothness. The Green–Schwarz mechanism is read as the technical completion of AX-2 (GGEP): canceling an anomaly is closing a snag in the composite gauge–gravity fabric, a geometric-smoothness condition rather than a quantization prescription.

Innovation 10 (I-BI6-10): BI-6 as the Stratum-2 anomaly theorem of the cohort. The result is logically prior to every downstream application:

$$\begin{aligned} \text{Constitution} &\Rightarrow \text{GG-6 stage} + \text{BI-6} \\ &\Rightarrow \text{local/global anomaly freedom} \xrightarrow{\text{OGGEP}} \text{lawful GDOS channel,} \end{aligned}$$

so Standard-Model closure, cosmology, and the life-science branches all rest on the anomaly theorem established here.

15 Predictions for falsification

A theoretical framework earns its place in physics not by being internally consistent alone — many consistent theories describe no aspect of the real world — but by making predictions that can be tested and, in principle, refuted. The BI-6/GDOS framework makes several such predictions, each carrying a global falsifier identifier F-BI6- k and tied directly to the OGN-5 arithmetic or the GG-6 gauge structure:

- (i) **(F-BI6-1) Proton stability.** Gauged $\text{U}(1)_B$ protects the proton: the observable falsifier is any confirmed $\Delta B = 1$ or $\Delta B = 2$ process (proton decay or $n-\bar{n}$ oscillation); only the electroweak-anomaly channel $\Delta B \in 3\mathbb{Z}$, which cannot deplete a single nucleon, is exempt. The protection is conditional on one named step — the holonomy computation that the descended boundary charge is baryon number on the split spectrum, stated and fenced in (Arisaka, 2026, A2-GG6) and carried as the boundary-side theorem of (Arisaka, 2026, A9-SM). With the $\text{SU}(5)$ coset vectors endpoint-decoupled by clause (C9) of Definition 6.1, no dimension-six proton decay is mediated at tree level at any scale, and the exempt channel remains the electroweak one.
- (ii) **(F-BI6-2) The 28 Standard Model parameters.** The companion paper (Arisaka, 2026, A9-SM) carries the 28 continuous Standard-Model parameters (6 quark masses, 3 lepton

masses, 3 CKM angles + 1 phase, 3 gauge couplings, the Higgs mass and VEV, the QCD θ angle, and the neutrino sector) from the OGN-5 arithmetic and the GDOS boundary projection, with an explicit per-parameter provenance triage (derived, derived-conditional, or asserted, each with a stated error budget). The falsifier is executable parameter by parameter: a measurement outside the budget GG-A9-SM states for a derived entry falsifies the framework.

- (iii) **(F-BI6-3) KK lock cluster at 3.85 TeV.** The Kaluza–Klein tower of the GG-6 throat is anchored at $M_{\text{lock}} \approx 3.85$ TeV, fixed by the OGN-5 throat depth $K = 35$ and the scale profile with no continuous parameter. Per (Arisaka, 2026, P1-KK), the HL-LHC falsifier is carried by the colored members of the cluster (the color octet), while the leptophobic Z'_B couples at the derived $g_B = 0.565$ (Theorem 5.8), its collider exposure quantified in (Arisaka, 2026, P1-KK); the mass anchor is theorem-level, the cluster morphology model-level.
- (iv) **(F-BI6-4) Kaxion dark matter near 8.1 μeV .** The Green–Schwarz tensor sector produces an axion-like particle (the “Kaxion”) with benchmark mass 8.1 μeV (≈ 1.95 GHz), as a *single* line: the chiral tensor leaves exactly one light pseudoscalar, and no partner tone exists in the assembled spectrum (the surviving decoupled vector mode couples to X_4 only through its massive Kaluza–Klein levels and yields no line). The canonical falsifier of (Arisaka, 2026, C1-Kaxion) is the closed window $\nu_K \in [1.59, 2.13]$ GHz, pre-registered against the seven loadings $\Xi_{\text{cap}} \in \{35, 38, 40, 42, 43, 45, 47\}$ of a ladder that companion grades as an ansatz, so that a null across the window counts regardless of the ladder’s later status. What a null kills is stated there at its grade, and this entry carries that grade: a completed scan of the window with no persistent line at the registered depth — a 90%-confidence exclusion $g_{90} \leq 0.8 g_{\text{DFSZ}}$ at every frequency in it, evaluated at the nominal local density $\rho_{\text{loc}} = 0.45 \text{ GeV cm}^{-3}$ with the standard-halo line shape — retires the calibrated placement of the line, the ladder that fences it, and the inflation-era pinning on which the single-patch history rests, sending the Kaxion to $\gtrsim 6$ GHz on the unpinned branch or the relic out of this sector; a confirmed compact-angle-like line outside the window retires the same objects (at $\gtrsim 6$ GHz, the pinning conjecture alone); a second light QCD-coupled line inside the window, at the first’s grade, refutes the count and names the light mode. None of these is a kill of the anomaly theorem or of the framework, and the dark-matter companion says so; the falsifier of *this* paper is the count — one light pseudoscalar, no partner tone — which the tensor sector fixes. The window lies in the reach of the haloscope program assembling around it — IBS-CAPP’s scan program from below, and the planned ADMX-EFR band (2–4 GHz) at its upper edge.
- (v) **(F-BI6-5) GUT-adjacent slow smooth inflation.** The inflationary phase is a GUT-adjacent plateau one e-fold below E_{GUT} , with $V_{\text{inf}}^{1/4} = 8.1 \times 10^{15} \text{ GeV}$, predicting the spectral tilt $n_s = 0.9634$ and the two-sided tensor-to-scalar ratio $r = 0.0038$ — the exact values at $N^* = 52.6$, of which 0.9620 and 0.0043 are the leading-order readings (Arisaka, 2026, A10-Cosmos). A measured n_s or r firmly outside these values — against the adjudicating dataset class and tolerances registered in (Arisaka, 2026, A10-Cosmos); current CMB

combinations differ among themselves at the level that decides this — would falsify the descent.

- (vi) **(F-BI6-6) Dark matter abundance.** The Kaxion must account for the full geometric inventory $\Omega_{\text{DM}} = 1/\pi - 1/(2\pi^2) \approx 0.27$ of (Arisaka, 2026, A10-Cosmos); the abundance mechanism is the consistency requirement carried in (Arisaka, 2026, C1-Kaxion), and a demonstrated failure of the Kaxion to saturate the inventory falsifies the dark-sector closure.
- (vii) **(F-BI6-7) The $U(1)_{Q_i}$ sector.** The informational gauge boson coupling to the $U(1)_{Q_i}$ current is a unique structural prediction of GG-6 with no analogue in the Standard Model or in any string compactification; its operational observables are carried by the boundary program of (Arisaka, 2026, A7-Qi4) (the Qi-4 pillars and the Q_B calibration), so it is listed here as a structural discriminator rather than a dated laboratory falsifier.

The scope of these predictions—from the proton lifetime to collider resonances to dark matter to inflation to the complete Standard Model parameter set—is the framework’s full exposure surface, and each item carries its provenance and its conditioning in the companion paper that owns it. The companion papers (Arisaka, 2026, A9-SM; Arisaka, 2026, A10-Cosmos; Arisaka, 2026, P1-KK; Arisaka, 2026, C1-Kaxion) develop the quantitative detail.

16 Cross-Paper Coherence Map

GG-A3-BI6 does not stand alone. The local and global anomaly-freedom theorems of Sections 5–6, the constitutional-admissibility theorem of Section 6.6, the model-level benchmark spectrum of Section 7, the trace-ratio and OGN-5 theorems of Section 9, and the GS-as-GGEP completion of Section 10.7 together feed into several companion papers of the GG-Theory program; conversely, the constitutional axioms and postulates that drive those theorems live in the companion paper GG-A1-Constitution, and the 6D bulk plus the BI-6 coefficient vector that the present paper takes as input live in GG-A2-GG6. The present paper sits at the *anomaly heart* of the program: GG-A1-Constitution supplies the constitutional clauses, GG-A2-GG6 supplies the geometric stage and the BI-6 four-form, GG-A3-BI6 closes the local and global quantum-consistency layer, GG-A5-GDOS carries the bulk-to-boundary projection that depends on GG-A3-BI6’s global theorem for its lawfulness, and GG-A4-Euclid, GG-A6-GRIP, GG-A7-Qi4, GG-A9-SM, GG-A10-Cosmos build the boundary-side phenomenology in their respective domains. This section states that structure in prose. The two supporting tables — which GG-A3-BI6 theorems are used by which companion papers, and which axiomatic inputs GG-A3-BI6 imports from GG-A1-Constitution and GG-A2-GG6 — are given with the dependency ledger in Appendix J, together with the audit trail that ties each pointer to a specific section or theorem of the cited companion.

17 Problems Solved, Open Questions, and Future Directions

GG-6 closes the bulk-side anomaly question, but the present paper does not close the GG-Theory program. Several of the questions raised by this paper are answered fully within the present text or

by named companion papers; several are settled in their core but await a specific technical sub-task; and a small number remain genuinely open even within the broader program. Honesty requires that we mark each one with its status. We follow the cohort’s triage convention: every entry O_k is labeled **Resolved** (the question has a complete answer in the present paper or in a named companion paper, with explicit citation), **Partially resolved** (the central claim is settled but a specific sub-task is still pending), or **Genuinely open** (no theorem, in this paper or any current companion paper, currently carries the answer). We address seventeen open problems below.

O1 (PS-BI6-1). Local anomaly cancellation of BI-6 in six dimensions

Resolved. The factorization $I_8 = \frac{1}{2}X_4^2$ with $X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - (2/5)F_B^2 - (2/5)F_{Q_i}^2$ is established in Theorem 5.1 (rank-one criterion) and Theorem 5.5 (BI-6 cancellation) of the present paper. The single-tensor Green–Schwarz counterterm $S_{\text{GS}} = \kappa \int B_2 \wedge X_4$ cancels the full perturbative one-loop anomaly via standard descent, and the BRST consistency of the resulting quantum gauge theory follows by Slavnov–Taylor identity. This is the local content of the central theorem of the paper.

O2 (PS-BI6-2). Global anomaly cancellation of BI-6 on the intended GG-6 background class

Resolved. Theorem 6.2 of the present paper proves that, on the nine-clause intended GG-6 background class, the combined phase of the chiral determinant and the exponentiated BI-6 counterterm is globally well-defined and gauge-invariant, using the modern Wu–Chern–Simons / Hopkins–Singer / Monnier–Moore differential-cohomology framework. The four hypotheses (H1)–(H4) of the global theorem are verified automatically on the intended class: the product / collar mapping tori bound spin eight-manifolds with extending data (H1), π_6 vanishes for $\text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$ (H2), the BI-6 lattice lift obeys the Wu-shifted integrality law (H3), and X_4 lifts to a global degree-four differential cocycle (H4). Hence BI-6 is completely anomaly-free at both the local and global levels on the physical intended class.

O3 (PS-BI6-3). Constitutional status of $\text{U}(1)_{Q_i}$: forced or admissible extension?

Resolved. Theorem 6.22 of the present paper (Section 6.6) settles this question. The legacy reading that “AX-3 + POS-4 + POS-1 forces $\text{U}(1)_{Q_i}$ ” is shown to overstate the axiomatic content: CPTP is a structural property of the GKLS channel (Lindblad 1976; Gorini–Kossakowski–Sudarshan 1976), not a covariance demand under a fresh $\text{U}(1)$. The constitutional clauses admit both the three-component bulk $G_{\text{Bulk}}^{(3)} = \text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B$ and the four-component bulk $G_{\text{Bulk}}^{(4)} = \text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$. The present paper adopts the four-component version on empirical grounds: the Arisaka equation $J_E/Q_B = J_e = J_B = J_{Q_i}$ requires $\text{U}(1)_{Q_i}$ on the boundary with bioenergetic conversion factor $Q_B \approx 0.1555$ eV per qubit matching ATP hydrolysis at biological temperatures; the OGN-5 dual-lock identity $K = LJ = M^2 + N^2 + L^2 = 35$ closes only for the four-component case under the equal-Abelian-treatment convention; and the boundary Q_i -4 pillars of GG-A7-Qi4 require $\text{U}(1)_{Q_i}$ as host symmetry. The companion paper Arisaka-GG-A2-GG6

(Arisaka, 2026, A2-GG6) cross-cites this resolution at its Theorem on Six Forces, at the BI-6 coefficient theorem, and at Objection 4 of its Common Objections section.

O4 (PS-BI6-4). Trace-ratio lock $|a/c| = |a/d| = 5/2$ from the OGN-5 ledger

Resolved. The trace-ratio identity that fixes the BI-6 coefficient vector $(a, b, c, d) = (1, -1, -2/5, -2/5)$ is established as the trace-ratio theorem in Section 9 of the present paper, and then used to derive the OGN-5 integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ as a Diophantine cascade. The integer 5 in the ratio $|a/c| = |a/d| = 5/2$ is the $L = 5$ size of the $SU(5)$ fundamental representation; the integer 2 is the rank of the equally-treated Abelian sector $U(1)_B \times U(1)_{Q_i}$. No continuous rescaling freedom remains; the ratio is rigid.

O5 (PS-BI6-5). Existence of a finite chiral spectrum saturating the BI-6 factorization

Resolved. Theorem 7.9 of Section 7 constructs an explicit benchmark spectrum — a modular core block plus an orbit family plus gravitational completion — that realizes the factorization $I_8 = \frac{\Omega}{2} X_4^2$ with all ten coefficient-matching conditions and all eight vanishing conditions satisfied. The benchmark proves that the structural theorem is satisfiable; minimality of the integer normalization Ω is the next sub-question (see O8 below).

O6 (PS-BI6-6). The 1984 Green–Schwarz mechanism as the technical pioneering of AX-2 GGEP

Resolved. Section 10.7 of the present paper carries the methodological reading that frames the 1984 GS achievement as the constitutional-precursor instance of the AX-2 (GGEP) Geometry–Gauge Equivalence Principle: GS demonstrated, in a specific 10D heterotic string model, that the metric and the gauge connection can sit together inside a single closed differential 4-form on the bulk manifold; AX-2 elevates that move from a technical refinement to a constitutional axiom. BI-6 is the four-component generalization of the two-component GS pattern, with the Lorentz, $SU(5)$, $U(1)_B$, and $U(1)_{Q_i}$ factors all entering on equal logical footing inside one closed gauge-invariant 4-form X_4 . Section 10.7 is the home of this reading; GG-A2-GG6 (Arisaka, 2026, A2-GG6) cross-cites it at its BI-6 coefficient theorem.

O7 (PS-BI6-7). Bulk-to-boundary projection through OGGE, with anomaly freedom as prerequisite

Resolved. The OGGE operational projection from the 6D bulk to the 4D boundary — whose lawfulness depends on the global anomaly freedom proved in the present paper — is the subject of GG-A5-GDOS (Arisaka, 2026, A5-GDOS). The anomaly proof of Sections 5–6 of the present paper supplies the prerequisite (Lemma on phase propagation through the OGGE channel of Section 6); GG-A5-GDOS then constructs the GDOS runtime as the boundary CPTP-channel image of the bulk and derives 4D quantum mechanics, 4D general relativity, and the low-energy Standard Model as

boundary projections. The handshake between the two papers is clean: BI-6 anomaly freedom in / GDOS runtime out.

O8 (PS-BI6-8). Minimal integer normalization Ω for the BI-6 spectrum

Partially resolved. Theorem 7.9 establishes *existence* of an integer Ω for which the BI-6 factorization $I_8 = \frac{\Omega}{2} X_4^2$ admits a finite chiral spectrum satisfying all eighteen coefficient-matching and vanishing conditions. The minimality sub-question — the smallest such Ω — is a bounded finite Diophantine search whose answer is not provided by the present paper. The companion paper GG-A10-Cosmos (Arisaka, 2026, A10-Cosmos) drives $\Omega \rightarrow 1$ dynamically as a cosmological attractor under the GUT-adjacent slow smooth inflationary descent, but the static minimal- Ω lower bound for the bulk Diophantine system itself is the open sub-task. The search is qualitatively simpler than the string landscape (it is bounded and finite), but it has not yet been carried out.

O9 (PS-BI6-9). Explicit Monnier–Moore mapping-torus partition functions for the non-supersymmetric BI-6 setting

Resolved (constructive). Subsection 6.4 now closes this problem at the level at which it was open: Theorem 6.13 proves the Wu–Chern–Simons refinement of the BI-6 tensor sector is unique (no discrete choice exists on a unimodular lattice), Lemma 6.17 excludes every deformation both structurally and constitutionally, the supersymmetry audit shows the Monnier–Moore construction consumes no supersymmetry, and Theorem 6.18 writes the partition function down — the canonical theta block of Eq. (71), single-valued by Theorem 6.9. The earlier status of this entry stands as history: the structural global theorem verifies the four hypotheses (H1)–(H4) of the framework automatically on the intended GG-6 class, so the global anomaly phase vanishes by hypothesis-checking rather than by explicit partition-function computation. An explicit numerical evaluation of the anomaly field theory’s partition function on each individual mapping torus — of the kind that Monnier and Moore carry out for $\mathcal{N} = (1, 0)$ supergravity — would serve as an independent confirmation of the structural argument; the same computation should treat the benchmark’s full chiral-tensor sector — the X_4 -coupled mediator together with its $4\Omega - 1$ gauge-singlet spectators — as a single anomaly field theory. On the intended GG-6 class the phase is now computed exactly (Lemma I.10: an APS index per species, and a vanishing filling signature for the tensor sector), and Theorem 6.9 now extends the structural computation to the winding and mixed-clutching classes as well, on signature-normalized fillings. With Subsection 6.4 in place, the explicit numerical evaluation guards no assumption at all: the refinement is unique, the characteristic is canonical, and the phase is one, each by theorem. The evaluation retains the standing of re-running a verified computation — an error-catching cross-check the 6D-anomaly community is welcome to perform, no longer a load-bearing open problem of this paper.

O10 (PS-BI6-10). Extension of the global theorem beyond the intended **GG-6** background class

Resolved for the **GG-6 gauge content; open only as a general classification program.** Theorem 6.9 (Subsection 6.3) extends the global theorem beyond the intended class: the reduced spin bordism groups $\tilde{\Omega}_5^{\text{spin}}(BG)$ and $\tilde{\Omega}_7^{\text{spin}}(BG)$ vanish for $G = \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$, every closed spin seven-datum bounds, fillings can be normalized to signature zero, and the phase is one — with no restriction on the base topology or on the homotopy class of the transformation, within the differential-cohomology accounting of Part II (whose defining refinement is now proved unique and whose partition function is constructed, Subsection 6.4). What remains open under this heading is not a gap in the **GG-6** chain but the community-wide question: the same analysis for *arbitrary* gauge packets and tensor lattices, where the bordism groups need not vanish and genuinely obstructed cases exist.

O11 (PS-BI6-11). Non-perturbative completion of the bulk theory

Genuinely open. Theorems 5.5 and 6.2 of the present paper establish, respectively, perturbative local and global anomaly freedom of BI-6 on the intended class. But a fully non-perturbative quantum theory of the 6D bulk — including the lattice (or other) regularization that defines the path integral, the global topology of the principal G_{Bulk} -bundle on M_6 , and the rigorous existence of the corresponding quantum field theory in the Wightman / Haag–Kastler axiomatic sense — is genuinely open. The relevant mathematical tools (constructive QFT, lattice gauge theory in dimension six, Osterwalder–Schrader reflection positivity for scale-profile backgrounds) are all in active development outside the **GG-Theory** program, and connecting them to the bulk theory is a long-term task.

O12 (PS-BI6-12). Generation count: why exactly three generations of fermions?

Genuinely open. The OGN-5 ledger of Section 9 supplies the integer $N = 3$ as the triplicity index, and the Quaternionic Goldilocks reading of GG-A2-GG6 (Arisaka, 2026, A2-GG6) suggests that three generations are the algebraically forced count for the $\text{SL}(2, \mathbb{H})$ rung of the magic Lorentzian-dimension ladder. But neither the present paper nor any current companion paper derives the three-generation structure of the boundary fermion content from the constitutional clauses as a theorem. GG-A2-GG6 lists this as O11 of its own Open Problems section and identifies three candidate mechanisms (topological zero-mode counting, magic-ladder algebraic forcing, Hosotani Wilson-line breaking on S_θ^1); none has been carried to a theorem. This is, in our view and in the view recorded by GG-A2-GG6, the single most important open theoretical problem of the **GG-Theory** program.

O13 (PS-BI6-13). Inflow consistency of the massive Kaluza–Klein baryonic vectors on the Dirichlet-terminated interval

Genuinely open. The Abelian boundary-condition clause (C9) of Definition 6.1 removes the zero mode of $A_{B,\mu}$, so no massless four-dimensional field carries a $U(1)_B$ gauge anomaly; the massive

Kaluza–Klein tower still couples to the anomalous boundary current. The registered computation is the explicit six-dimensional check that the Green–Schwarz term, acting through the tensor’s massive modes, renders that coupling consistent level by level on the terminated interval. What is being checked has an ordinary Kaluza–Klein name: each massive level of $A_{B,\mu}$ takes its longitudinal state from the corresponding mode of $A_{B,r}$, so the tower is Stueckelberg-massive level by level, and this problem asks whether the inflow bookkeeping of that pairing closes. Inflow makes this the expected outcome, and the level-by-level calculation is standard for interval compactifications; the check has not been carried out here, and until it is, the consistency of the massive baryonic tower is an expectation, not a theorem.

O14 (PS-BI6-14). The tensor remnant’s canonical decay constant

Genuinely open. The mode analysis of Section 5.9 leaves three inputs undetermined: the orientation of the anti-self-duality condition relative to the warp (which selects between the two kinetic norms and moves the answer by e^K), the transverse cell scale, and the tensor coupling. For the infrared-weighted chirality with a Planckian cell the canonical decay constant lands near the TeV scale, which is excluded; the value the companion kaxion paper calibrates from the relic closure corresponds to a descent depth near $K/2$. Which is interesting, and is not proof. Until this computation closes, the remnant’s decay constant — and with it the absolute frequency inside the window fenced by K — is a calibration, not a derivation.

O15 (PS-BI6-15). The seam’s gauge kinetic term

Genuinely open. A terminated boundary carries whatever terms the termination induces, and for a gauge field the lowest-dimension one is a kinetic term on the seam, $-\frac{\rho}{4}\delta(r-K)F_{\mu\nu}F^{\mu\nu}$, with $\rho_K = \rho k$ its size in curvature units so that ρ_K/K is its share of the throat’s kinetic weight. The registered question is whether the Dirichlet termination of clause (C9) fixes ρ_K , and at what value. The question matters because it is the one escape from GG-A9-SM’s oblique tension that the constitution can answer on its own terms (Arisaka, 2026, A9-SM): with the electroweak first level held at the lock, a seam term changes the infrared condition to $f'_n(K) = \rho_K x_n^2 f_n(K)$, reduces the tower sum only mildly — by twelve percent at $\rho_K = 5$, one seventh of the throat’s weight — but makes one mode light, $x_1 = 0.667$ against 2.450. Holding that mode at the lock raises the curvature unit to 5.77 TeV, a factor 5.7, and lifts every other level with it: the second electroweak level to 23 TeV, any Neumann tower without the term to 14 TeV, and the spin-2 and the baryonic vector — the latter blind to a seam kinetic term, being terminated by a Dirichlet condition — to 22 TeV. On that placement the oblique parameter is $T = 0.111$ for a brane Higgs, 0.067 at $\Delta_H = 4$, 0.031 at $\Delta_H = 2$, inside or at the edge of the allowed band; the light mode’s infrared amplitude falls from 70 to 6.6, so every lock-cluster rate that scales with the infrared overlap falls by an order of magnitude. The mechanism is the known one (Carena et al., 2003, Davoudiasl, Hewett, and Rizzo, 2003). *What decides it:* the termination theorem, if it fixes ρ_K of order five, closes the oblique tension as a prediction of this paper and moves the spin-2 anchor and the baryonic vector off the lock by a factor 5.7 — which the collider companion’s falsifier would then have to be re-registered against; if it fixes zero, the tension stands at kill grade exactly as GG-A9-SM prints it. Either way

the question has moved from precision data to the constitution, and it is registered here so that it cannot be mistaken for a survival. The numbers are from a released script and are reproduced in GG-A9-SM's Open Problem O22.

O16 (PS-BI6-16). The interval-line period under Dirichlet termination

Genuinely open. With $A_{B,\mu}$ Dirichlet at both ends, the interval Wilson line $\int_0^K A_{B,r} dr$ is gauge-invariant (Section 5.9), and its background value is a datum of the admissible class this paper has not computed. One reading of the cosmological residue ζ_Λ — the twist-rate reading, $\zeta_\Lambda = \frac{1}{2}\langle A_{B,r} \rangle = \pi/(LK)$ by an unweighted mean over the interval, registered in the cohort's open-item ledger as C-S3-1 and adopted by no cohort paper — is a conjecture whose falsifier lives here: it holds only if the interval-line period is $\int_0^K A_{B,r} dr = 2\pi/L$ under Dirichlet termination. *What decides it:* the period, computed on the admissible class. A period of $2\pi/L$ keeps the conjecture alive and does not by itself establish it, since it must also meet GG-A10-Cosmos's rider on the residue rule — the $1/K$ must come from the winding count of the dual lock, not from the length of the interval (Arisaka, 2026, A10-Cosmos); any other period retires the reading. GG-A10-Cosmos's Hubble rows do not move on this problem in either direction; they move on the residue computation of GG-A5-GDOS's Open Problem O6 (Arisaka, 2026, A5-GDOS).

O17 (PS-BI6-17). The sign of the family modes' winding

Genuinely open. Proposition 5.10 fixes the anti-self-duality of the tensor; it says nothing about the sign of the θ -winding of the family modes on the family stratum. That no negative windings occur on the cell — the statement that makes the family stratum's floor a point rather than a manifold, and its landscape a perfect Morse function with one floor — is at present an input of GG-A6-GRIP (Arisaka, 2026, A6-GRIP), not a theorem of this one. *What decides it:* a derivation of the winding sign from the six-dimensional chirality of the family spinors, which the anomaly fixes, would make it a theorem of this paper and remove an input from GG-A6-GRIP. Until then it is registered here with two owners, this paper and GG-A6-GRIP, and GG-A6-GRIP's uniqueness statement on the family stratum carries it as a hypothesis.

Summary of the triage. Of the seventeen open problems above, eight are **Resolved** (O1–O7 and O9), two are **Partially resolved** (O8 and O10), and seven are **Genuinely open** (O11–O17). The eight Resolved problems are answered either by theorems of the present paper (O1–O6 and O9) or by a named companion paper that builds on this paper's anomaly proof (O7). The two Partially resolved problems each name the specific sub-task that is still pending — the minimal- Ω Diophantine search (O8), and the extension of global consistency from the GG-6 gauge content to a general classification (O10). The seven Genuinely open problems — non-perturbative completion (O11), generation count (O12), the terminated-interval inflow check (O13), the remnant's decay constant (O14), the seam's gauge kinetic term (O15), the interval-line period (O16), and the winding sign (O17) — carry the framework's theoretical research agenda forward. The test of an honest framework is not that it answers every question on the first pass, but that it knows which questions it has answered, which it has half-answered, and which it has not yet touched.

Future directions. The triage above sets the near-term research program in order of leverage. (i) *Global phases beyond the intended class*: the winding and mixed-clutching transformations of Open Problem O10 are the one place where a genuinely new Dai–Freed evaluation is required; the intended-class phases are now computed exactly (Lemma I.10). (ii) *The completion sector’s spectrum*: the gauge-singlet fermions admit Dirac pairings that can lift most of the sector, while the spectator tensors, all of one chirality, cannot pair — their boundary projection and cosmological exposure are the sector’s real open physics. (iii) *The holonomy-descent theorem*: the single fenced step behind F-BI6-1, that the descended boundary charge is baryon number on the split spectrum — the subject of the dedicated proton-stability treatment. (iv) *The termination’s bookkeeping, continued*: the mode analysis of Section 5.9 has identified the termination as a boundary condition rather than a shift, and the variational-principle question becomes two registered computations — O13, the explicit six-dimensional check that the Green–Schwarz term, acting through the tensor’s massive modes, renders the massive Kaluza–Klein baryonic vectors’ coupling to the anomalous current consistent on the Dirichlet-terminated interval; and O14, the tensor remnant’s canonical decay constant (the chirality’s orientation relative to the warp, the cell scale, the tensor coupling), which decides whether the Kaxion’s decay constant is derived or remains calibrated. Three further questions the termination raises are registered beside them: the seam’s gauge kinetic term (O15), the interval-line period (O16), and the winding sign of the family modes (O17). (v) *The minimal- Ω search* (O8) and, above all, (vi) *the generation count* (O12) — still, in our view, the single most important open problem of the program.

Scope and limitations. Before we close, let us state the scope and the limits of the present paper as a self-contained block. *Scope.* This paper proves complete local and global anomaly freedom of the modified Bianchi identity BI-6 on the physical intended GG-6 background class, using the modern Wu–Chern–Simons / Hopkins–Singer / Monnier–Moore differential-cohomology framework. The local theorem (Theorem 5.5) is the standard Green–Schwarz descent argument, applied to the four-component $X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - (2/5)F_B^2 - (2/5)F_{Q_i}^2$. The global theorem (Theorem 6.2) verifies the four hypotheses (H1)–(H4) of the differential-cohomology framework automatically on the intended class. The constitutional-admissibility theorem (Theorem 6.22) shows that the GG-Theory Constitution admits both the three-component and four-component bulks; the four-component version is preferred on empirical grounds. The model-level benchmark spectrum theorem (Theorem 7.9) constructs an explicit finite chiral spectrum saturating the BI-6 factorization. The OGN-5 theorem of Section 9 derives the integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ as a Diophantine cascade. The GS-as-GGEP completion subsection (Section 10.7) frames the 1984 Green–Schwarz mechanism as the technical pioneering of AX-2 (GGEP). *Limits.* The paper does *not* carry out the bulk-to-boundary projection that produces 4D quantum mechanics, 4D general relativity, and the low-energy Standard Model — this is the subject of GG-A5-GDOS (Arisaka, 2026, A5-GDOS). The paper does *not* reconstruct the universal constants $\{c, G_N, \hbar; k_B, Q_B\}$ from the bulk geometry — this is the subject of GG-A4-Euclid. The paper does *not* carry out the empirical calibration of the boundary phenomenology (CKM, neutrino masses, fine-structure constant, bioenergetic Q_B , cosmological inventory) — these are the subjects of GG-A7-Qi4 (Arisaka, 2026, A7-Qi4), GG-

A9-SM (Arisaka, 2026, A9-SM), GG-A10-Cosmos (Arisaka, 2026, A10-Cosmos). The paper does *not* determine the minimal integer normalization Ω of the BI-6 spectrum (Open Problem O8), nor provide a non-perturbative completion of the bulk theory (Open Problem O11); the explicit partition-function evaluation (O9) and the extension beyond the intended class (O10) are now carried by Subsection 6.4 and Theorem 6.9 respectively, at the grades their registry entries state. *Dependencies.* The validity of every theorem in the present paper is conditional on the correctness of the constitutional clauses AX-1–AX-4 and POS-1, POS-3, POS-4, POS-5, POS-6 of GG-A1-Constitution (Arisaka, 2026, A1-Constitution). What the present paper takes from GG-A2-GG6 (Arisaka, 2026, A2-GG6) is the *object* and not the theorems: the six-dimensional stage, the four-factor gauge packet, the rank-one tensor channel and the BI-6 four-form enter here as clauses (C1)–(C4) of the admissible class of Definition 6.1, which is background data quantified over, not a premise imported from a proof. Every proof below is carried out inside that class and consumes nothing further from any companion. The two papers are therefore parallel consumers of the constitution rather than a chain: GG-A2-GG6 establishes that the constitution *selects* this bulk, the present paper that the bulk so specified is *consistent*, and neither derivation runs through the other. Should the constitutional clauses be revised, the chain of theorems must be re-walked; should the specification of the object change, the admissible class must be restated and the proofs re-run inside it. The full inward map of dependencies is recorded in Section J.2.

Where the discussion leaves things. Three summaries are worth carrying into the Conclusions.

The first is about what kind of result this is. The anomaly theorem is not a prediction; it is a license. It says the six-dimensional stage is a consistent place to do physics, and it says so without consuming a single measured quantity. Everything the cohort builds on that stage — the boundary projection, the Standard-Model codebook, the cosmological inventory — would be void if this Part had come out the other way, and none of it is made true by this Part coming out the way it did. A license is a necessary condition, not a sufficient one, and the paper is careful not to trade on the difference.

The second is about where the framework can be killed. Seven falsifiers are listed, and each names a measurement, a value, and a precision. Three of them will be tested in the present decade: the resonance cluster near 3.85 TeV, the kaxion line near 8.1 peV — predicted to stand alone, with no companion tone — and the inflationary parameters. The others are longer-term. What the framework does not have is a way to survive any of them failing — there is no continuous parameter to adjust and no landscape to move within. That is a genuine risk and it is taken deliberately.

The third is about what remains undone, and it is the part a reader should weigh most heavily. Twelve open problems are triaged. Two are genuinely open: a non-perturbative completion of the bulk theory, and the generation count. The second of these is the one that matters most for how this framework should be judged, because the number three appears in the ledger as a color count and appears in nature as a generation count, and the paper does *not* derive the second from the first. It says so in four places, and it says so here again, because the temptation to blur the two is exactly what an honest framework has to resist. Anyone who can close that gap — or show it cannot be closed — would be doing the most useful work available on this program, and the author would welcome the collaboration.

Part VI

Conclusions

What has been proved, what it means, and why it matters — in plain language first, then in the precise register of the theorem chain, and last the place of the result in the series

This Part says what has been established, and it says it twice: once in ordinary language, for a reader who has not followed the calculations, and once in the register of the theorem chain, for a reader who has.

The order is deliberate. A result that cannot be stated plainly is usually a result that is not yet understood, and the discipline of saying it without symbols is a check on the author as much as a service to the reader. Section 18 is that statement. It uses no notation and assumes no background beyond a first course, and a reader who reads only that section will come away with an accurate picture of what this paper does and does not claim.

Section 19 then restates the same content precisely: what was proved, what it rests on, which claims are conditional and on what, which remain open, and where the result sits in the twenty-two-paper cohort. Section 20 closes with the wider reading — what it would mean, if the argument holds, for the relationship between geometry and the forces — and marks clearly where that reading stops being a theorem and starts being a proposal.

18 The result in plain language

Here is what this paper proves, said as plainly as we can say it.

Every force we know is described by a gauge theory, and every gauge theory built from chiral matter carries a hidden risk: the quantum theory can betray the very symmetry that defines it. When that happens the theory does not become approximate — it dies. Probability leaks, ghosts fail to decouple, and no measurement it predicts can be trusted. Physicists call this failure an *anomaly*, and passing the anomaly test is the entrance examination that every candidate theory of nature must sit.

GG-Theory proposes that the world we see is the four-dimensional face of a six-dimensional geometry. This paper sat that geometry down for the examination and graded every question. The result: it passes — completely. The one-loop failure is canceled exactly by a single tensor field flowing in from the bulk; the subtler global failures, the ones that only show up when the theory is carried around non-contractible loops in the space of its own backgrounds, cancel too — first on the class of backgrounds the theory actually uses, and then, once the topological fine print is removed, on every shape of space the question can be asked about; and the passing grade is not generic. The theory passes because five small integers — 1, 3, 5, 7, 35 — take exactly the values they take, and it would fail for almost any other choice.

That is the deep physical intuition worth carrying away. In this framework the anomaly is not a quantum disease to be cured; it is a classical fact about the geometry — a statement, descended from Bianchi’s 1902 identity, that curvature is stitched together smoothly — and the quantum loop is merely the instrument on the boundary that measures it. Canceling the anomaly is not surgery; it is the recognition that gravity and the gauge forces were cut from one cloth in the bulk, so precisely that the seams — the coefficients $(1, -1, -2/5, -2/5)$ — cannot be re-sewn in any other pattern. The examination is passed not by tuning but by rigidity.

And rigidity is testable. The same integers that pass the anomaly test put numbers on the sky and in the laboratory: an inverse fine-structure constant $\alpha^{-1} = 137$ at leading order — the number that says the electric force between two electrons is about 137 times weaker than the crude quantum unit — a dark-matter note near 1.95 GHz, predicted to stand alone in its fenced window, a resonance cluster near 3.85 TeV, a proton that does not decay. If any of those numbers comes out otherwise, the geometry that passed this examination is the wrong geometry — and the paper you are holding tells you exactly which page fails. Few theories offer their critics that courtesy; we consider it the point.

19 What has been established, what it rests on, and what remains open

19.1 The four ingredients under one roof

The central achievement of this paper is a complete local-and-global anomaly theorem for BI-6 on the physical intended GG-6 background class. The theorem does more than restate the Green–Schwarz idea in new symbols. It is a concrete six-dimensional result with four distinct ingredients brought under one roof: the explicit BI-6 factorization

$$X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - \frac{2}{5} F_B^2 - \frac{2}{5} F_{Qi}^2, \quad I_8 = \frac{1}{2} X_4^2,$$

the descent proof of the local anomaly cancellation, the Wu–Chern–Simons / differential-cohomology completion of the tensor counterterm, and the automatic admissibility theorem for the intended GG-6 category.

A second achievement is that the constitutive integer packet

$$(M, N, L; J, K) = (1, 3, 5; 7, 35)$$

is now stated and used at theorem level. In particular, the paper makes clear that $N = 3$ is fixed first as the bulk color triplicity index by the BI-6 trace ratio, that $L = 5$ is the defining unified block size of SU(5), that $J = 7$ is the minimal symmetric closure cardinality of the charge packet $\{0, \pm 1, \pm 2, \pm 3\}$, and that $K = 35$ is both the rectangular lock index and the quadratic witness. This is the arithmetic lock that keeps the BI-6 sector finite and sharply constrained.

A third achievement is conceptual. Global anomaly freedom is shown to be a prerequisite for a lawful OGGEF/GDOS bulk-to-boundary map. If the six-dimensional parent theory were only

locally consistent but globally multi-valued, then the projected boundary functional would inherit an unphysical topological phase ambiguity. The present theorem therefore does more than repair a bulk gauge theory. It opens the door to the entire boundary program.

With this summary in hand, we can now place the paper in the broader context of the series.

19.2 The place of this paper in the series

This paper should be read as the direct mathematical continuation of GG-A1-Constitution (Arisaka, 2026, A1-Constitution) and GG-A2-GG6 (Arisaka, 2026, A2-GG6). GG-A1-Constitution proposed the GG-Theory Constitution as a compact admissibility system. GG-A2-GG6 derived the six-dimensional GG-6 stage and wrote down BI-6 as the structurally distinguished modified Bianchi identity. The present paper asks the next unavoidable question: does that stage define a consistent quantum theory? The answer given here is yes, on the physical intended GG-6 class, and the proof is explicit.

In that sense the derivation chain now reads

$$\boxed{\begin{aligned} \text{GG-Theory Constitution} &\implies \text{GG-6 stage} + \text{BI-6} \\ &\implies \text{complete local/global anomaly freedom} \xrightarrow{\text{OGGEP}} \text{lawful GDOS boundary channel.} \end{aligned}} \quad (123)$$

The later applications to Standard-Model closure, cosmology, chemistry, life, and mind depend on this chain, but the anomaly theorem established here is logically prior to all of them.

19.3 What is proved, what is suggested, and what remains open

Theorem-level statements in this paper are category-precise. BI-6 is completely anomaly-free on the physical intended GG-6 class. OGN-5 is theorem-level inside that same class. The explicit off-diagonal cancellations, the affine orbit family, and the chirality reinterpretation of the apparent negative multiplicity are all closed. The constitutional admissibility of $U(1)_{Q_i}$ as an optional fourth Abelian factor of the bulk gauge group is settled in Section 6.6 (Theorem 6.22). The methodological reading of the 1984 Green–Schwarz mechanism as the technical pioneering of AX-2 (GGEP) is settled in Section 10.7.

Two kinds of statements are intentionally separated from that theorem core. First, the stronger claim for every arbitrarily enlarged non-supersymmetric six-dimensional category is not proved here and should not be advertised as a corollary. Second, the late-time operational normalization $\Omega_{\text{eff}} \rightarrow 1$ is given here as a condensed GDOS attractor sketch rather than as the main anomaly theorem itself. The purpose of that sketch is self-contained logical clarity, not inflation of the anomaly claim. These distinctions are not weaknesses. They are part of the paper’s design principle: to keep theorem, corollary, and downstream dynamical application separated cleanly enough that each can be judged on its own terms.

The explicit triage of every open problem raised by the present paper — with each entry O_k labeled **Resolved** (in the present paper or in a named companion paper), **Partially resolved** (with a specific sub-task pending), or **Genuinely open** (no theorem in any current paper carries

the answer) — is given in Section 17, the closing section of Part V, immediately preceding these Conclusions.

20 Big picture and closing reflection

A higher-dimensional theory becomes physically meaningful only when three layers close simultaneously: geometry, gauge law, and global quantum consistency. GG-6 supplies the six-dimensional geometric stage. BI-6 supplies the locked gauge-gravity tensor identity. The present theorem supplies the global quantum consistency. Only after all three layers are closed is it legitimate to project the theory operationally to the four-dimensional world.

This is why anomaly freedom matters so much in the axiomatic program. It is not a technical footnote appended to a geometrical construction. It is the sharpest filter that nature imposes on any candidate quantum theory in higher dimensions. The Green-Schwarz mechanism taught the string community this lesson in ten dimensions in 1984; the six-dimensional anomaly-polynomial program deepened it through the work of Sagnotti, Schwarz, Kumar, Morrison, Taylor, Monnier, Moore, and many others. The present paper applies the same lesson — with the same mathematical tools — to a framework that departs from the string tradition in its physical assumptions while respecting its mathematical standards.

There is a broader point worth making. The history of theoretical physics suggests that the most productive ideas are not always those that are most popular at the time of their introduction. The original Green-Schwarz anomaly cancellation result of 1984 was initially met with skepticism by parts of the particle physics community; it took years for its full implications to become clear. The six-dimensional anomaly program itself was a relatively quiet corner of string theory until the modern differential-cohomology and bordism-theoretic tools matured. The GG-Theory axiomatic approach is new, unfamiliar, and comes from outside the string-theory tradition. It should be judged not by its provenance but by its mathematical content, its internal consistency, and its falsifiable predictions.

The five integers (1, 3, 5; 7, 35) encode all dimensionless normalization freedom in the BI-6 identity. They are not chosen to fit data; they are derived from the gauge architecture of the six-dimensional bulk. The same integers later reappear in the derivation of Standard Model parameters, in the KK mass scale, in the cosmological hierarchy, and throughout the GDOS boundary program. Whether this arithmetic compression reflects a deep truth about nature is, ultimately, an empirical question. But the fact that it passes the anomaly-cancellation test — the same test that selected $SO(32)$ and $E_8 \times E_8$ in ten dimensions — is a non-trivial piece of evidence.

If the theorem established in this paper survives further scrutiny, then GG-6 will stand as more than an attractive geometric picture: as a finitely locked and quantum-mechanically admissible parent theory. The observed four-dimensional world, with its three generations of fermions, its specific gauge couplings, its particular Higgs vacuum, and its peculiar pattern of masses and mixings, would then be not an accident to be fitted but a necessary consequence of a constitutional law operating in six dimensions. Whether that vision is correct is for experiment and further mathematical analysis to decide. The present paper has taken one step — proving that the six-dimensional parent theory

is consistent — and the road ahead is open.

The anomaly theorem proved in the present paper is, in this constitutional reading, the strongest result the framework requires at Stratum 2. The further constitutional content of AX-3 and AX-4 — the operational bulk-to-boundary CPTP channel, the QM postulates as theorems, the dissolution of the measurement problem, the unique OGN-5 ledger, the elimination of the string landscape — is the subject of the companion papers GG-A5-GDOS, GG-A6-GRIP, GG-A7-Qi4, GG-A8-QM, GG-A9-SM, GG-A10-Cosmos, and GG-A11-Origin. Each of those companion contributions rests on the Stratum 2 anomaly consistency proved here. In that sense, the present paper closes the gating prerequisite for the entire cohort program: the seven cohort papers downstream of GG-A3-BI6 are each, in their different ways, applications of AX-3 and AX-4 on a six-dimensional bulk whose Stratum 2 consistency is the subject of the present BI-6 theorem.

A last word on how to read this. The strongest claim in this paper is a consistency theorem, and consistency theorems are modest things. They do not say a theory is true. They say it is not disqualified — that one particular way of being wrong has been ruled out. What makes this one worth writing down is that the way of being wrong is severe, the test admits no adjustable parameters, and the same structure that passes the test hands over five integers that turn into numbers an experiment can check.

Whether those numbers are right is not something this paper can settle, and it should not be read as trying to. The proper response to a framework of this kind is neither belief nor dismissal but attention to the short list of things it forbids. If a resonance cluster near 3.85 TeV is excluded, if the kaxion line is not where the ledger puts it, if the proton decays, then the geometry that passed this examination is the wrong geometry, and the paper has said in advance which page fails. That is the whole of the offer.

What remains open is stated plainly and is not small. The generation count is not derived. The bulk theory has no non-perturbative completion. Several of the sharper questions raised here — the minimal spectrum problem, the extension of the global theorem beyond the intended class, the relation between the integer ledger and the F-theory lattice conditions — are well-posed mathematical problems that this paper does not solve. They are stated in the hope that someone will, and the author would be glad of the company.

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All scientific concepts (especially the foundation of the GG-Constitution, GGEP/GG-6/BI-6, and OGGE/ GDOS/Gi-4/Qi-4 Pillars), theoretical judgments, physical interpretations, and any remaining errors are the sole responsibility of the author.

Part VII

For the Curious Reader

Twenty-five questions a thoughtful reader without a physics degree actually carries into a paper like this one — answered plainly, and in the order they usually arrive

The Conclusions have said what the paper proves. This Part answers the questions a reader is entitled to ask next, and it assumes no physics beyond a first course. It is placed here, and not at the back, because a reader who wants these answers should not have to walk through sixty pages of bordism theory to reach them. Nothing in the Part is needed by the technical argument; nothing in the technical argument is needed to read it. The specialist objections — the ones a particle theorist or a differential-cohomology specialist raises — are a separate matter and are answered in their own Part, further on.

The twenty-five questions were chosen the way a teacher chooses them: they are the ones that actually get asked. Several are hostile, and they are printed hostile. One of them — why a demand for internal consistency should be entitled to forbid anything at all in a world under no obligation to be mathematically comfortable — is the hardest question anyone has put to this framework, and the answer to it concedes more than it settles.

Two habits are followed throughout. Where a claim rests on a step that is not closed, the answer says so and names the step, rather than rounding it up to a result. And where the honest answer is that the framework does not reach the question, the answer says that too. A reader who finishes this Part should be able to state not only what the paper claims but where it stops, which is the more useful of the two.

Nothing here is a summary of the proof. The proof is Part II, and it is not summarizable in this register without becoming false. What these pages give instead is the shape of the claim: what an anomaly is, why anything has to cancel, where a handful of whole numbers could come from, and what would have to be measured for the whole construction to be abandoned.

21 The big questions, plainly answered

The twenty-five questions that follow are not a specialist's. They are the ones a thoughtful reader without a physics degree actually carries into a paper like this one: what an anomaly is and why anything should have to cancel, where a handful of whole numbers could possibly come from if nobody measured them, why a demand for internal consistency is allowed to predict the weight of a star, and whether an integer that cannot be adjusted is a strength or simply a hostage. Several of these have been settled in popular science in directions this paper does not take, and a reader is owed plain answers before the specialist objections begin. They are met first, and plainly.

The hardest of the twenty-five is the fifth — why a consistency requirement — a statement about mathematics — should be entitled to forbid anything at all in a world that was under no obligation

to be mathematically comfortable. The response concedes the difficulty rather than dissolving it, and says which part of it the paper can carry and which part it cannot. Where a claim rests on a step that is not closed, the response says so and names the step; where the honest answer is that the framework does not reach the question, it says that too.

Question 1 (Q-BI6-1) (on the curious reader)

“What is an ‘anomaly’, in plain terms, and why does anything have to cancel?”

Response. An anomaly is a symmetry that survives the classical description of a system and dies when the quantum description is written down. That is the whole of it, and the word is badly chosen — nothing anomalous in the everyday sense is happening.

Here is why it matters. Some symmetries in physics are decorative: if they fail, a quantity you thought was conserved turns out not to be, and you adjust. Others are structural — they are what makes the description consistent at all, because the theory is written in terms of quantities that are only defined up to a choice, and the symmetry is what guarantees the choice does not matter. If one of those dies quantum-mechanically, the theory does not become slightly wrong. It becomes meaningless: probabilities stop summing to one, and the same calculation gives different answers depending on an arbitrary convention.

So “must cancel” is not an aesthetic preference. It is the requirement that the description be a description. What makes this powerful is that the cancellation condition is an equation relating the contents of the theory — what kinds of matter there are, what charges they carry — and equations of that kind have solutions rather than accommodating anything you like.

The historical demonstration is worth knowing. In 1984 Green and Schwarz showed that in ten dimensions the anomaly cancels for exactly two gauge groups out of the enormous space of candidates. That single calculation redirected a field. The argument in this paper is of the same type, run in six dimensions.

Question 2 (Q-BI6-2) (on the curious reader)

“Why should nature care about a number like 35? It seems completely arbitrary.”

Response. It would be arbitrary if it had been chosen, and the reason to think it was not is that it arrives with four other numbers, all determined together, none of them adjustable.

The way to see this is to notice what would happen if 35 were wrong. It is not a coefficient sitting in front of a term where a different value would give a slightly different prediction. It participates in a closure condition alongside the others, and the condition either closes or it does not. Change the 35 and nothing in the framework closes — there is no repaired version with a different number, because the numbers are a solution rather than a parameterization.

It also helps to see where such numbers come from generally, because they are less exotic than they look. Integers appear in physics wherever something has to fit: the energy levels of an atom, the number of ways a lattice can be arranged, the winding of a loop around a hole. In none of those cases did anyone choose the number. It is what a counting problem returned. The ledger here is the same kind of object — the answer to a counting problem posed by the arena’s structure.

The fair form of your suspicion is not that 35 looks odd. It is that a framework producing a

specific integer should be asked what else that integer commits it to, so that the number is exposed rather than merely stated. Here it fixes the strength of the electric force at low energy (the number 137), fences the narrow window that contains a dark-matter line near 1.95 gigahertz, and fixes the mass of a cluster of new particles near 3.85 TeV. All three are measurable this decade — one by instruments already running, the others by instruments now being assembled. What the ledger does *not* fix is the number of families of matter — the paper is explicit that the three colors it derives and the three families nature shows are not the same three, and that connecting them is an open problem it does not solve.

Question 3 (Q-BI6-3) (on the curious reader)

“Where do these whole numbers come from if nobody measured them?”

Response. From a counting problem that the arena poses and that has one answer.

It is worth being concrete about the shape of such an argument, because “derived from geometry” is the kind of phrase that can mean anything. The structure of the arena determines what kinds of twisting are possible on it. The consistency requirement demands that certain contributions from those twistings sum to zero. That is a system of equations in whole numbers — how many of this kind, how many of that — and such systems are famously rigid: most have no solutions at all, and those that do usually have exactly one. Here there is one.

Nothing in that chain involves a measurement, and nothing in it involves a choice. It also means the numbers are not free to move if a measurement disagrees. That is the whole point, and it is the property being bought at the price of no longer having anywhere to retreat to.

The honest limit is that the argument is conditional on the arena. Six dimensions, a specific symmetry group, and the constitutional requirements are all inputs, established elsewhere in the cohort and not derived here. Given those, the ledger follows and its uniqueness is proved. Deny those, and the whole calculation is about nothing. A skeptical reader is better served attacking the arena than attacking the integers, because the integers are the easy part.

Question 4 (Q-BI6-4) (on the curious reader)

“What actually breaks if the ledger is wrong?”

Response. Nearly everything downstream, and it is worth naming the pieces so the answer is not just an alarming adjective.

The value of the electric force at low energy follows from the ledger — the number 137 is $2^7 + 3^2$, and the 7 and the 3 are two of the five integers — so if the ledger is wrong that agreement is a coincidence and the framework’s most-cited result is gone. The window that fences the dark-matter line near 1.95 gigahertz and the particle cluster near 3.85 TeV descend from the same integers; a different ledger gives different numbers and there is no version of the framework that gives no number. The relationships among charges — which is why charge comes in the multiples it does — follow from the same source. And the constitutional claim that the framework has no free parameters depends entirely on the ledger being forced rather than fitted, so if it were adjustable the central methodological boast would be false.

This is not a comfortable answer and it should not be softened. A framework in which every piece is independently replaceable is a framework in which nothing carries weight. The ledger

carries a great deal, which is why the paper spends its length proving uniqueness rather than asserting it, and why a reader wanting to find a fatal error should look there first.

What does *not* break is worth noting too. The arena itself, its dimension and its symmetry group are upstream and survive — they would simply no longer force the physics they are claimed to force, which would leave a mathematically respectable structure with nothing to say about the world.

Question 5 (Q-BI6-5) (on the curious reader)

“Why should a demand for internal consistency be allowed to predict anything about the world?”

Response. This is the hardest question here and it does not have a clean answer. Saying so is better than managing it.

The worry is exact. Consistency is cheap. One can write down any number of internally impeccable structures and almost all of them are about nothing. From the fact that a framework hangs together, nothing follows about whether nature implements it. Any argument of the form “it must be so, or the mathematics would not close” is, taken by itself, an argument about mathematics.

What can be claimed is narrower. Consistency is not being used here to establish that the world is a certain way; it is being used to *eliminate*. Given an arena that is assumed rather than derived, the closure requirement admits one ledger instead of a family of them. That is a statement about a search space. Whether nature is in the search space at all is settled by measurement and by nothing else, and the framework’s entire empirical exposure lives in that gap.

There is a historical reason to take the method seriously without mistaking it for proof. Anomaly cancellation — exactly a demand for internal consistency — has repeatedly narrowed the field of admissible theories in particle physics, and the narrowing has survived contact with data. The Green–Schwarz calculation is the famous case. That is an encouraging record and it is not an argument. The accurate summary is that consistency is a filter of unusual reach and no kind of evidence at all, and a reader who keeps those two facts apart will read this paper correctly.

Question 6 (Q-BI6-6) (on the curious reader)

“You say a handful of whole numbers fixes what happens in a particle detector. Why should a detector care about six-dimensional algebra?”

Response. It should not care, and the framing hides what is actually being claimed. The detector is not responding to the algebra. It is responding to a small extra term in the equation that governs how a dense object balances its own weight against its internal pressure — and the size of that term is what the algebra fixes.

The chain is short enough to state. The consistency condition forces a specific depth for the extra dimension, and a depth is a mass. It is not a mass anyone chose: it is the ledger integer 35, used once. The mass comes out near 3.85 TeV. The value is not adjustable, because it is fixed by the same whole numbers that had to be whole for the theory to exist at all.

What makes this a good prediction rather than an impressive-sounding one is the absence of

freedom. There is no coupling to tune, no scale to choose. The framework says the mass is a specific value, with a specific pattern of companion states around it.

And it is currently measurable. The High-Luminosity LHC reaches that energy, and a second prediction — a dark-matter line between 1.59 and 2.13 gigahertz — lies in the reach of the haloscope program now assembling around it. A null result at better than the predicted precision is not a setback to be absorbed; it ends the ledger, and with it the paper.

Question 7 (Q-BI6-7) (on the curious reader)

“Isn’t anomaly cancellation just bookkeeping about infinities? That sounds like a technicality, not a discovery.”

Response. The bookkeeping reputation is real and it is a misreading of what the calculation does.

The reputation comes from renormalization, where infinities genuinely are absorbed by a procedure that many physicists found distasteful for decades and that Dirac never accepted. Anomaly cancellation is a different operation and the difference is worth seeing. Renormalization takes a quantity that came out infinite and relates it to a measured one. Anomaly cancellation does not absorb anything: it is a condition that either holds or fails, given the contents of the theory, and when it fails there is no procedure that rescues the theory.

That is why it constrains. A procedure that can be applied to anything tells you nothing about what exists. A condition that most candidate theories fail tells you a great deal — and the overwhelming majority of ways one might imagine assembling matter and forces do fail it. The Standard Model passes, and the passing is delicate: the charges of the quarks and leptons enter a sum that cancels only because the values are what they are. This is standard physics and not a claim of this framework.

What this paper adds is running the same requirement one level up, on the arena rather than within a fixed four-dimensional theory, and finding that the condition again admits almost nothing. Whether that step is legitimate is a technical question with a technical answer, and it is the right place for a skeptical reader to spend effort.

Question 8 (Q-BI6-8) (on the curious reader)

“If the ledger is unique, why did it take until now to find it?”

Response. Because the question it answers was not being asked in a form that had an answer, and the reason for that is more interesting than the delay.

For most of the twentieth century the project was to write down a four-dimensional theory matching the measurements, and the parameters that theory needed were treated as things to be measured rather than derived. That is not a failure of imagination; it is the correct strategy when you have data and no organizing structure, and it produced the Standard Model, which is the most accurate theory anyone has ever had. Nobody asks where twenty-eight numbers come from while they are still being measured.

Asking the question in a form with an answer requires committing to a specific arena first, and that commitment was not available. The algebraic argument that bounds the dimension is old mathematics, but connecting it to the constitutional requirements used here is not, and without

both halves the counting problem cannot even be posed.

It should also be said plainly that lateness is weak evidence in either direction. Long-unnoticed results do exist and so do long-unnoticed errors, and the fact that something was not found before tells you almost nothing about whether it is correct now. The reason to take the ledger seriously is that it is exposed — a coupling strength, a particle mass, a charge pattern — and not that it is new.

Question 9 (Q-BI6-9) (on the curious reader)

“Why five numbers rather than one? A theory of everything ought to need one.”

Response. It ought not, and the intuition that it should is worth examining because it has misled people.

There is no principle saying a fundamental description must be specified by a single quantity, and nothing in the history of physics supports the expectation. What matters is not how many numbers appear but whether they were chosen. Five integers that are the unique solution of a closure condition constitute one fact, not five. Five continuous parameters fitted to data constitute five decisions. The count of symbols is a poor guide to the count of assumptions.

There is also a structural reason the number here is not one. The ledger describes how several different kinds of structure fit together on the arena — how many of each kind of twisting, at what relative weights — and a single number could not carry that information any more than a single number could specify the shape of a crystal. What is being reported is a pattern, and patterns have several entries by nature.

The version of your question that does bite is this: could a different arena have produced a different five-number solution, so that the uniqueness is only uniqueness-given-the-arena? Yes, and the paper says so. The arena is an input. What is proved here is that given it, there is no second ledger — which is a real result and a conditional one, and conflating the two would be the error.

Question 10 (Q-BI6-10) (on the curious reader)

“Charge comes in whole multiples of a smallest unit. Is that the same phenomenon?”

Response. It is, and noticing that is the fastest route into why any of this should be believed. The quantization of charge is one of the oldest unexplained facts in physics. Every electron carries exactly the same charge as every other, to the limits of measurement, and every charge ever measured is a whole multiple of a common unit. Nothing in the Standard Model explains this. The values are put in by hand, and their exact equality — the reason atoms are neutral to better than one part in 10^{21} — is an unexplained coincidence of the deepest kind.

Geometry supplies a natural mechanism, and it is old: if charge counts how many times something wraps around a closed internal direction, then it is a whole number for the same reason that the number of times a rope winds around a post is a whole number. There is no continuous path from three windings to four. Dirac saw a version of this in 1931 and it has been part of the subject ever since.

What this framework contributes is not the mechanism but the specifics: which windings the arena permits, and therefore which charges can exist. That converts a qualitative explanation

into a list, and a list can be wrong. If a particle turns up with a charge outside the permitted set, the account fails — which is a considerably better position than an explanation that predicts only that charges are quantized without saying which.

Question 11 (Q-BI6-11) (on the curious reader)

“What is a ‘Bianchi identity’ and why does yours need modifying?”

Response. An identity in this sense is a statement that is true automatically — not a law imposed on the world but a consequence of how the quantities were defined, in the way that the statement that the boundary of a boundary is empty is true of any shape whatever. The standard Bianchi identity says that a certain combination of field quantities vanishes identically, and it is the reason magnetic charge does not appear in ordinary electromagnetism.

Modifying an identity sounds like a contradiction and is not. What is modified is not the mathematics but which quantity the identity applies to. When the arena carries additional structure, the natural field quantity is not the one you first write down; there is an extra piece, and the identity holds for the corrected object rather than the naive one. The corrected statement is again automatic once the correct object is identified.

Why this matters is that the extra piece is not free. Its form is fixed by the requirement that the anomaly cancel, and its coefficients are the ledger. So the modification is where the geometry enters the physics: the identity is the equation, the correction term is the unknown, and the closure condition is what solves it.

It is worth adding that this kind of modification is not exotic. It is exactly the structure Green and Schwarz used in ten dimensions in 1984, and the mechanism has been standard since. What is claimed here is its consequence in six, where the counting comes out differently.

Question 12 (Q-BI6-12) (on the curious reader)

“Is a shift of a fraction of a percent in stellar weight a real prediction, or is it lost in the noise?”

Response. It is at the edge of what is currently measurable, which is the uncomfortable and honest answer.

The predicted shift is a fraction of a percent, and the framework states the fractional offset at which a null result counts. No single star settles it; a population has to be weighed. That is a real limitation and no amount of framing removes it.

What makes the prediction testable anyway is that the shift is *coherent*: it acts in the same direction, by the same fractional amount, for every compact star. Random measurement error does not do that. A population of stars measured independently can therefore reveal a systematic offset far smaller than any individual uncertainty, in the same way that averaging many noisy measurements of a fixed quantity converges. Instruments now operating are accumulating exactly such a population, and the precision of the ensemble is approaching the required level. The risk this creates should be stated. If the population study returns a null result at better than the predicted precision, the ledger is wrong, and there is no adjustment available — the size of the shift is a ratio of integers. A framework whose most accessible prediction sits just at

the edge of current capability is a framework about to find out, which is the position a theory should want to be in and an uncomfortable one to occupy.

Question 13 (Q-BI6-13) (on the curious reader)

“Why do integers show up in physics at all? Nature isn’t obviously counting anything.”

Response. It is counting rather often, and once you start noticing it is hard to stop.

The electron in an atom does not occupy arbitrary energies but a discrete ladder, and the reason is that a wave has to fit around a closed path a whole number of times. A crystal admits a small finite number of symmetry types and no others, which is why there is no five-fold periodic lattice. The number of ways a loop can wind around a hole is an integer because there is no continuous path from one winding number to another. In none of these cases is anything counting in the sense of keeping a tally. The integer is the answer to a question about fitting or about topology. What all of them share is a structure with a constraint that admits only certain configurations, and this is why integers are a signature of geometry rather than of arithmetic. Where you find one in physics, there is usually a shape underneath.

The claim in this paper is of that type: the ledger integers are the answer to a fitting question posed by the arena, and their whole-numberedness is not a coincidence but a consequence of the question having that form. Which is also why they are the framework’s most exposed feature. An integer cannot be nudged, and a framework that has been forced onto integers has given up the ability to survive a disagreement by a small adjustment.

Question 14 (Q-BI6-14) (on the curious reader)

“What is the difference between ‘forced’ and ‘fitted’ if both end with the same number?”

Response. The difference is what happens next, and it is the only difference that matters.

A fitted number is answerable to the measurement it was fitted to and to nothing else. If a second measurement disagrees, the number moves. It will always be possible to make it agree with whatever is in front of it, which is why a fitted number, taken alone, is not evidence of anything — it is a record of what was measured, written in the theory’s notation.

A forced number is answerable to everything it touches. It was produced by a condition that knew nothing about the measurement, and it now determines other quantities that were not used to obtain it. If any one of those disagrees, there is no move available.

So the test of whether a number in a framework is forced or fitted is not to examine how it was obtained — accounts of derivation can be constructed after the fact, and have been. The test is to ask what else it commits the framework to, and whether those commitments were made before the checking. Here the ledger fixes a coupling strength, a charge pattern and two particle masses, none of which was used in deriving it, and all of which are on the record in advance. That is the whole of the defense, and it is checkable rather than rhetorical.

Question 15 (Q-BI6-15) (on the curious reader)

“Could a different arena give a different unique ledger? Then what have you proved?”

Response. Yes, and stating that plainly is more useful than the theorem is.

What is proved here is conditional: given this arena — six dimensions, this symmetry group,

these constitutional requirements — the closure condition admits exactly one ledger. Change the arena and the counting problem changes, and one would expect a different answer or none. The uniqueness is uniqueness *within* a specified setting, not across all possible settings, and a reader who took it for the latter would be taking more than is on offer.

What that conditional buys is still worth something. It means the framework has no internal freedom: having accepted the arena, nobody can adjust anything to accommodate a measurement. All the freedom is concentrated in one place, at the top, in the choice of arena — which is declared, is a small number of statements, and is argued for on separate grounds in the companion that establishes it.

It also tells a critic exactly where to attack. There is no point looking for slack in the ledger; there is none. The argument to break is the one that fixes the arena, and if it breaks, everything here becomes a correct calculation about a structure nature does not use. That is a clean place for a disagreement to live, and clean is the most one can ask of a framework's exposure.

Question 16 (Q-BI6-16) (on the curious reader)

“What would a failed anomaly cancellation actually look like, physically?”

Response. It would not look like a small discrepancy. It would look like the theory failing to be a theory, and that is worth making concrete because “inconsistent” is an abstract-sounding verdict.

In a theory with an uncanceled anomaly of this kind, the probabilities computed for a set of mutually exclusive outcomes do not sum to one. That is not a prediction that is slightly off; it is the absence of a prediction, because the numbers being computed have stopped being probabilities. Related symptoms follow: the same physical process gives different answers depending on a choice that was supposed to be arbitrary, and quantities that must be positive come out negative.

This is why the condition is treated as a filter rather than as a hypothesis. One does not test whether a theory with an uncanceled anomaly describes the world; such a theory does not describe anything. It is discarded before any measurement is considered.

The physically interesting statement is the contrapositive, and it is the one this framework uses: since the world manifestly does have consistent probabilities, whatever describes it must satisfy the condition — and the condition is restrictive enough that satisfying it is a real constraint on what can exist. That is the step from an abstract requirement to a definite spectrum, and it is the paper's central move.

Question 17 (Q-BI6-17) (on the curious reader)

“Isn't our universe just one of countless others, and we happen to be in the one where the numbers work?”

Response. It might be. What this paper offers is a reason the question may not need the answer. The multiverse addresses a specific problem: the constants look arranged, and if they could have been otherwise the arrangement demands explanation. An ensemble supplies one. That argument is coherent and is made by careful people.

Its premise is that the constants could have been otherwise. Here they could not — not because

of a philosophical objection to variation, but because the closure condition has one solution and there is no second ledger to be drawn. If that is right, the observation that motivated the ensemble is no longer surprising and the ensemble has nothing to explain.

It is worth adding the asymmetry that makes this checkable rather than a matter of preference. The ensemble picture makes no prediction about the strength of the forces, or about which charges can exist, or about where a new particle must be. This one does. Whichever is right, only one of them is in a position to be embarrassed by a measurement, and that is the more useful kind of theory to have even if it turns out to be the wrong one.

Question 18 (Q-BI6-18) (on the curious reader)

“The universe looks fine-tuned for life. Doesn’t that point to a designer?”

Response. It points to something requiring explanation, and the move from there to a designer needs a premise that this framework denies.

The observation is genuine: small changes in certain quantities and there is no chemistry, no structure, nothing that lasts. Feeling the pull of design when confronted with that is not naive. The inference requires that the quantities could have been set differently. A dial that was turned implies a hand on it. If the values are the unique solution of a closure condition, there was no dial, and asking who chose them is like asking who chose the ratio between a circle’s circumference and its diameter.

None of this argues against a designer, and it should not be recruited to. Someone who believes on other grounds is untouched by this paper, and so is someone who does not. What changes is one particular argument, which has carried a great deal of popular weight and which loses its premise if the ledger is forced. Whether it is forced is settled by a haloscope scan and a collider search, which is a strange place for a theological argument to be decided and is nonetheless where this one now sits.

Question 19 (Q-BI6-19) (on the curious reader)

“If the integers are fixed, is my future already written? What happens to free will?”

Response. Nothing here touches your future, and the inference does not hold even though it feels tight.

What is fixed is the rule book: what charges are permitted, how strong the forces are, what new particles must exist and where. What is not fixed is any particular history within it. The distinction is between the rules of chess and the game played last Tuesday, and a rule book admitting no alternative says nothing about which of an enormous number of permitted games occurred.

There is a sharper point underneath. Even a fully deterministic rule book delivers no written future in any sense that could trouble anyone, because the only thing capable of computing what the rules imply for a system as complicated as a person is a system at least that complicated, running at least that long. There is no shortcut, no vantage point, and no book. A universe can be lawful all the way down and still be, in the only sense available to anything inside it, unwritten until it happens.

And the limit: this paper is about a cancellation condition and its solution. What agency is, and whether the concept survives contact with physics at all, is not a question it reaches.

Question 20 (Q-BI6-20) (on the curious reader)

“Why is there something rather than nothing?”

Response. This framework does not touch it, and pretending otherwise would be a sales pitch. Everything here is conditional — given an arena with certain properties, the closure condition has this solution. The question of why there is an arena sits outside the construction entirely, and no rearrangement of the argument reaches it. That is not a gap further work will close; it is a different kind of question.

What can be contributed is narrower and is not nothing. If the framework is right, the space of possible somethings is far smaller than it appears: not a landscape of coherent worlds from which one was drawn, but very nearly a single admissible structure. That converts “why this, out of all there might have been?” into “why anything?” — which is harder and cleaner. Narrowing a question is a service and is not an answer.

The temptation is to say something about nothingness being unstable or the question being ill-posed. Both are available, both have serious defenders, and neither follows from anything in this paper.

Question 21 (Q-BI6-21) (on the curious reader)

“A handful of integers explaining this much makes me suspicious. Shouldn’t it make you suspicious too?”

Response. It should and it does, and Eddington is the name to keep in mind. He was among the finest astrophysicists of his generation and spent his last years deriving the fine-structure constant and the number of particles in the universe from combinatorial arguments that convinced almost nobody. His reputation never recovered. Anyone deriving integers is one bad step from that. What separates a derivation from a coincidence is not the quality of the agreement. It is whether the numbers came from machinery that exists for another reason, and whether they commit you to anything you did not already know. A number reverse-engineered from a measurement will always match it.

Here the ledger comes out of a cancellation condition on an arena fixed beforehand by an argument that knows nothing about detectors, and it then fixes a coupling strength, a charge pattern and two particle masses that were not used to obtain it. That is the only defense available and it is the right one: not “too good to be chance” but “here are three further consequences, go and look.”

Keep the suspicion. If the consequences fail, the numbers were a coincidence, and no elegance in their derivation will save them.

Question 22 (Q-BI6-22) (on the curious reader)

“Why should I take one person’s framework seriously against thousands of physicists working on other approaches?”

Response. Not because of who wrote it, and not despite it either — both are the same error

with opposite sign.

The practical question is fair, since attention is finite. The useful filter is exposure, not headcount. A framework supported by thousands that forbids nothing measurable is a worse use of your attention than one supported by a single person that forbids specific things and names the instrument that will decide. Community size tracks funding history and fashion, and does not track correctness.

There is a genuine cost to working outside a large community. Errors are found faster where more people are looking, and a framework developed largely by one person carries a higher chance of containing a mistake a hundred readers would have caught in a week. Internal coherence does nothing to reduce that.

So do not take it on trust. Take the compact-star prediction, which needs no six-dimensional mathematics to evaluate — a fractional shift of a stated size, in a stated direction, across a population — and ask whether it is stated precisely enough to be wrong. If it is not, ignore the rest. If it is, that deserves attention regardless of whose framework it came from.

Question 23 (Q-BI6-23) (on the curious reader)

“What would convince you that you are wrong?”

Response. Four things, named rather than gestured at.

A sweep of the microwave window between 1.59 and 2.13 gigahertz finding no dark-matter line, at the coupling depth the framework states, would end the ledger. A proton decay would do the same by a different route, since the protection follows from the same gauge structure and there is no parameter to adjust. A particle carrying a charge outside the permitted pattern would break the link between the arena’s structure and what can exist. And an internal failure counts fully: if the uniqueness proof has a hole, the claim that the ledger is forced rather than chosen is gone whatever the instruments say, and that is the most likely of the four.

What would not convince: a disagreement in a regime the framework does not claim to cover, or the failure of a result whose hypotheses are already flagged as open. A falsifier list is only honest if it excludes things, and a reader should be wary of any proponent unwilling to say which failures are survivable.

The framework’s exposure is unusually concentrated. Almost everything rests on the ledger, so almost everything can be ended by one measurement. That is a feature and it is also a reason to check the uniqueness argument before checking anything else.

Question 24 (Q-BI6-24) (on the curious reader)

“Why should mathematics be able to forbid a universe?”

Response. It cannot, and the framing is where the trouble starts. Mathematics forbids nothing about the world. What it can do is tell you that certain *descriptions* are unavailable — and since we have no access to the world except through descriptions, that turns out to be almost as useful and is not the same thing.

The distinction matters here. Nothing in this paper says the universe could not have had four families of matter. It says that no consistent description of the kind under consideration produces four. If nature has four, then the kind of description under consideration is not the

right kind, and the framework is wrong rather than nature being illegal. That is the correct direction of the inference and it is easy to state backwards.

Physics has been in this position before and the outcome has gone both ways. The requirement that a description be relativistically consistent forbade a great many candidate theories and the survivors turned out to be right. The requirement that a description be built on absolute simultaneity forbade a great many candidate theories and was itself wrong. There is no way to tell in advance which case one is in.

What follows is a discipline rather than a reassurance: a framework arguing from consistency owes an unusually clear account of what it forbids, so that if nature turns out to be outside its reach the fact is discovered quickly rather than argued about.

Question 25 (Q-BI6-25) (on the curious reader)

“I’m not a physicist and I’ve got this far. What is the one thing you want me to take away?”

Response. That the interesting question is not whether a framework is true but whether it has arranged to be caught, and that this one has arranged it unusually tightly.

Almost everything written about fundamental physics for a general audience — in both directions, enthusiastic and debunking — argues about whether a set of ideas is right. That argument is rarely settleable from an armchair and it is not what moves science. What moves it is the duller question of whether a claim has been put in a form some measurement could contradict, and whether that was said beforehand.

This paper takes a requirement that any consistent description must satisfy, applies it to a specific arena, and finds that it admits exactly one set of whole numbers. Those numbers then say which charges can exist, how strong the electric force is at low energy, and where two new particles must be. Instruments now running will settle the last of those within a few years.

So take away a habit rather than a belief. When you next meet a grand claim about the universe — this one included — ask what it forbids. If the answer is nothing, or keeps changing, you have learned what you needed without following any mathematics. If the answer is a short list of things somebody is about to measure, that claim is worth your attention whether or not it survives.

Where to go from here. A reader who has come this far without the mathematics has, in fact, the substance of the paper. The claim is that a consistency requirement in six dimensions, applied without any freedom to tune, forces five small whole numbers, and that those numbers turn into quantities that can be measured. Everything before this Part is the demonstration that the requirement really is a requirement and that the numbers really are forced.

If you want to go further, the shortest useful route is not through the proof. It is through Section 15, which lists what would end the framework, and through Section 17, which lists what it has not managed to do. Between them those two sections say more about how much confidence a framework deserves than any summary of its successes could.

If you want the argument itself, Part VI states the conclusions precisely, and Part II is the proof. If a specific technical objection occurred to you while reading these answers, Part IX may already

have it — the fifty-nine there are the ones specialists raise, written in their language, and several of them are sharper versions of questions answered plainly above.

Part VIII

Technical Appendices

A graduate-level course in six-dimensional anomaly cancellation, read side by side with string theory — from-scratch derivations of the descent formalism, the ten-dimensional Green–Schwarz mechanism and its six-dimensional dictionary, the rank-one criterion, the OGN-5 arithmetic, the benchmark spectrum, the gravitational anomaly polynomials, and the global (Dai–Freed) completion — preceded by the reference dictionary and followed by the theorem checklist

The body of this paper has emphasized the theorem chain. The ten appendices that follow supply the detailed technical content behind each step, and taken together they are intended to be read as *a graduate-level course in anomaly cancellation seen through GG-Theory and through string theory side by side*, addressed to an incoming graduate student in particle physics. We organize them into two groups. The *reference* appendices (A and J) provide the abbreviation dictionary and the theorem checklist matrix, and serve as the reader’s quick lookup. The *constructive* appendices are the course proper: the complete descent proof, sector by sector (B); the ten-dimensional Green–Schwarz mechanism worked from scratch, with the dictionary that translates every one of its moves into the six-dimensional setting of this paper (C); the rank-one criterion and all eighteen coefficient-matching and vanishing conditions (D); the OGN-5 integers derived from BI-6 (E); the benchmark spectrum — core block, Diophantine orbit system, and gravitational completion (F–H), the last with the gravitational anomaly polynomials themselves derived from the index theorem; and the global anomaly proof on the admissible class (I), opened by a self-contained primer on the eta invariant and the Dai–Freed formalism. Where the material exists in the string-theory tradition, each appendix treats it in *three steps*: first the standard machinery, from scratch, in the form a graduate student meets it in the textbooks; then a frank statement of how string theory deploys that machinery and what its deployment assumes; and finally the GG-6 derivation, in which the same machinery runs on the six-dimensional bulk with the constitutional clauses doing the work that supersymmetry or worldsheet consistency would have done.

Throughout these appendices we adhere to two stylistic commitments, following the cohort convention. First, every derivation is presented step by step with no skipped intermediate manipulations: a graduate student should be able to reproduce each calculation with paper and pencil. Second, the formal steps are accompanied by *physics intuition* paragraphs that translate the algebra into a mental picture. The combination is designed to make the technical material usable as a course while preserving the rigor of the main text.

Prerequisites and recommended textbooks. The course assumes the standard first-year graduate sequence in quantum field theory and is meant to be read *alongside* the canonical texts, not in place of them. The recommended path through each appendix is: read the matching textbook

treatment first, return here for the from-scratch recap, and then follow the GG-6 derivation and the string-side comparison. For self-study we particularly recommend the following, each cited again in the appendix to which it is best matched:

- **Anomalies, first pass.** [Peskin and Schroeder \(1995\)](#) (Ch. 19) and [Weinberg \(1996\)](#) (Vol. II, Ch. 22) — the triangle anomaly, the Fujikawa measure, and anomaly matching as a graduate student first meets them.
- **Anomalies, in depth.** [Bilal \(2008\)](#) and [Harvey \(2005\)](#) — the two standard lecture-note treatments of the descent formalism, anomaly polynomials, and inflow; and [Bertlmann \(1996\)](#) and [Fujikawa and Suzuki \(2004\)](#) — the two book-length treatments, the first cohomological, the second path-integral.
- **Geometry and topology background.** [Nakahara \(2003\)](#) — characteristic classes, index theorems, and the eta invariant at the level this paper uses them; [Milnor and Stasheff \(1974\)](#) for the underlying characteristic-class theory.
- **The string side.** [Green, Schwarz, and Witten \(1987\)](#) (Vol. II, Ch. 13) — the original textbook treatment of the ten-dimensional Green–Schwarz mechanism; [Polchinski \(1998b\)](#) (Ch. 12) and [Becker, Becker, and Schwarz \(2007\)](#) (Ch. 5) — the modern treatments; [Ibáñez and Uranga \(2012\)](#) — the phenomenological bridge; and [Taylor \(2011\)](#) — the six-dimensional supergravity landscape this paper’s Part IV compares against.
- **Group theory for the spectrum computations.** [Slansky \(1981\)](#) and [Georgi \(1999\)](#) — indices, Casimirs, and branching rules for the SU(5) representation data of Appendix F.
- **The global (nonperturbative) side.** [Witten \(2016\)](#) — fermion path integrals, the eta invariant, and anomaly inflow in the modern reflection-positivity language; and [García-Etxebarria and Montero \(2019\)](#) — Dai–Freed anomalies deployed on particle-physics gauge groups, the closest published analogue of the checks run in Appendix I.

Throughout, GG-A1-Constitution–GG-A11-Origin, GG-P1-KK–GG-P6-Proton, and GG-C1-Kaxion–GG-C3-BH refer to the cohort papers, to which the GG-6-side steps repeatedly point.

A Summary of Abbreviations

This appendix is the full dictionary: every abbreviation used in this paper, together with the program-defined names of the companion papers it cites and of the acronym plate below, together with the companion paper that introduces each one. The handful of names that appear before they are defined are glossed in the Notation block of the front matter; this is where the rest live. All program-defined names are rendered in *sans-serif type* (GG-Theory, GG-Constitution, GG-6, GDOS, BI-6, OGN-5, QGL, Gi-4, Qi-4, GRIP, BFFT, etc.) so the reader can see at a glance which terms belong to the GG-Theory program proper. Standard physics abbreviations imported from the wider literature (KK, RS, QFT, TOV, etc.) are rendered in plain uppercase Roman, following the

convention of the major physics journals. Mathematical objects are in roman or italic in the usual way.

Table 12: Every abbreviation used in this paper, together with the program-defined names of the companion papers it cites and of the acronym plate below, each with its meaning and the place it is introduced in full. Typography carries information here: program-defined names are set in sans serif, standard physics abbreviations in plain uppercase Roman, and mathematical objects in roman or italic as usual, so a reader can tell at a glance whether a name belongs to this framework or to the general literature. The third column links each entry to its primary source — the present paper for names defined here, the companion paper otherwise. A reader who meets an unfamiliar name anywhere in the volume should come here first.

Symbol	Meaning	Primary link
<i>Framework name and core program objects</i>		
GG-Theory	Grand Geometric Theory (also: Geometry-Gauge Equivalence Theory)	GG-A1-Constitution, Preface
GG-Constitution	The constitutional charter of the GG-Theory program: four axioms (AX-1–AX-4) and seven postulates (POS-1–POS-7)	GG-A1-Constitution, §4–§5
GG-6	Geometry-Gauge locking in six dimensions: the 6D bulk	GG-A2-GG6, §12 (T1-GG6)
GDOS	Geometric Dynamics of Open Systems: the boundary law implied by OGGE	GG-A5-GDOS
BI-6	Green–Schwarz-modified Bianchi identity at 6D (the principal object of the present paper)	GG-A3-BI6, §5
OGN-5	Occam-Gauge-Normalization 5-integer ledger ($M, N, L; J, K$) = (1, 3, 5; 7, 35)	GG-A3-BI6, §8–§9
QGL	Quadruple Gauge Locking: the four-factor bulk gauge group $\text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Qi}$	GG-A2-GG6, §19 (T8-GG6)
Gi-4	Geometric-Invariance four-pillar structure (TGG, CCI, GFF, GRIP)	GG-A6-GRIP
Qi-4	Quantum-Information four-pillar structure (Qi-QUA, Qi-EP, Qi-CON, Qi-HAL)	GG-A7-Qi4
<i>Four axioms (AX-1–AX-4)</i>		
AX	Axiom (axiomatic primitive; non-negotiable)	GG-A1-Constitution, §1.5
LC	AX-1 Locality and Causality	GG-A1-Constitution, §4.2
GGEP	AX-2 Geometry–Gauge Equivalence Principle	GG-A1-Constitution, §4.3
OGGE	AX-3 Operational/Open Geometry–Gauge Equivalence Principle (boundary observation dictionary)	GG-A1-Constitution, §4.4
MSMF	AX-4 Maximum Symmetry, Minimum Freedom (no hidden continuous knobs)	GG-A1-Constitution, §4.5
<i>Seven postulates (POS-1–POS-7)</i>		
POS	Postulate (admissibility filter; testable in principle)	GG-A1-Constitution, §1.5
GI	POS-1 Gauge invariance across scales (LGI / DGI / GGI)	GG-A1-Constitution, §5.3
UEP	POS-2 Unified Extremal Principle (PLA $\Leftrightarrow$ PLT / MEPP / FEP)	GG-A1-Constitution, §5.4
AF	POS-3 Anomaly Freedom and cancellability by inflow (the constitutional clause central to the present paper)	GG-A1-Constitution, §5.5
CP	POS-4 Closure and Positivity (CPTP-admissible open dynamics)	GG-A1-Constitution, §5.6

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Symbol	Meaning	Primary link
RG	POS-5 Renormalization-Group covariance (scale covariance)	GG-A1-Constitution, §5.7
OGN	POS-6 Occam Gauge Normalization (canonical integer/rational normalization; <i>formerly</i> MRHN)	GG-A1-Constitution, §5.8
MF	POS-7 Measurement and Falsifiability (operational contract + kill criteria)	GG-A1-Constitution, §5.9
LGI	Local Gauge Invariance (microscopic / closed-system layer of GI)	GG-A1-Constitution, §5.3
GGI	Global Gauge Invariance (macroscopic / ledger layer of GI)	GG-A1-Constitution, §5.3
<i>Gi-4 pillars</i>		
TGG	Total Geometric–Gauge object (Pillar I of Gi-4)	GG-A6-GRIP
CCI	Covariant Conservation Identity (Pillar II of Gi-4)	GG-A6-GRIP
GFF	Geometric Free-Energy Functional (Pillar III of Gi-4; <i>absorbs the legacy</i> EEGG ledger)	GG-A6-GRIP
GRIP	Gauge-Runtime Invariance Pipeline (Pillar IV of Gi-4)	GG-A6-GRIP
GEP	Geometry–Gauge Energy Principle (energetic block of GFF; <i>formerly</i> EGG)	GG-A6-GRIP
CEP	Curvature–Entropy Principle (entropic block of GFF)	GG-A6-GRIP
DGI	Dynamical Gauge Invariance (initial stage of GRIP)	GG-A6-GRIP
GSSB	Geometric Spontaneous Symmetry Breaking (middle stage of GRIP)	GG-A6-GRIP
GA	Geometric Attractor (final stage of GRIP)	GG-A6-GRIP
GGA	Ground Geometric Attractor (the unique deepest GA; where the runtime halts)	GG-A6-GRIP def:GGA; GG-A11-Origin
GRIP-In	Fixed-landscape regime of the runtime (descent halting at the GGA; consumes the BI-6 action budget)	GG-A6-GRIP
GRIP-Out	Landscape-reorganizing regime (supplies the next descent’s initial datum)	GG-A6-GRIP
<i>Qi-4 pillars</i>		
Qi-QUA	Quantum information for QGL (Quadruple Gauge Locking) – Pillar I of Qi-4	GG-A7-Qi4
Qi-EP	Quantum information for the UEP (Unified Extremal Principle) – Pillar II of Qi-4	GG-A7-Qi4
Qi-CON	Quantum information for Conformal Invariance – Pillar III of Qi-4	GG-A7-Qi4
Qi-HAL	Quantum information for the Holographic Attractor Lattice – Pillar IV of Qi-4	GG-A7-Qi4
OH	Outer Hologram (boundary projection of bulk data; outward face of EIC)	GG-A7-Qi4
IH	Inner Hologram (inner self-modeling layer; inward face of EIC)	GG-A7-Qi4
EIC	Evolving Inner Cosmos (boundary completion of Qi-4 as the OH+IH composite)	GG-A7-Qi4
<i>Qi-HAL acronyms</i>		
MePMoS	Memory–Prediction–Motion Sensing	GG-A7-Qi4, Qi-HAL
NHT	Neural Hologram	GG-A7-Qi4, Qi-HAL
HAL	Neural Holographic Tomography	GG-A7-Qi4, Qi-HAL

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Symbol	Meaning	Primary link
Q-HAL	Quadratic / Cubic HAL	GG-A7-Qi4, Qi-HAL
BFFT	Binary Fourier / Fractal Fourier Transform	GG-A7-Qi4, Qi-HAL
<i>GDOS-internal acronyms (program-defined)</i>		
SIP	System Instance Packet (canonical GDOS input unit)	GG-A5-GDOS
SeOS	Selfish Open System	GG-A5-GDOS
<i>Program-defined mathematical symbols</i>		
Q_B	Quantum-information Bit – the bioenergetic Arisaka constant $Q_B \approx 0.1555$ eV per qubit; matches ATP hydrolysis at biological temperatures	GG-A2-GG6, §19 (T8-GG6)
X_4	Bulk anomaly four-form $X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - (2/5)F_B^2 - (2/5)F_{Q_i}^2$ (the central object of the BI-6 cancellation)	GG-A3-BI6, §5
H_3	Three-form field strength of the two-form tensor potential B_2 , $H_3 = dB_2 + \text{CS}$ (Chern–Simons three-forms); modified Bianchi identity $dH_3 = X_4$	GG-A3-BI6, §5
<i>Standard physics abbreviations (plain Roman, journal convention)</i>		
<i>Open-system / quantum-channel formalism</i>		
CPTP	Completely Positive Trace-Preserving (channel admissibility for open evolution; AX-3/POS-4)	—
GKLS	Gorini–Kossakowski–Lindblad–Sudarshan (master equation for Markovian CPTP semigroups; POS-4)	—
GNS	Gelfand–Naimark–Segal (construction)	—
NZ	Nakajima–Zwanzig (memory-kernel projection for non-Markovian open dynamics)	—
POVM	Positive Operator-Valued Measure (generalized measurement model; AX-3/POS-7)	—
<i>Quantum and classical foundational theories</i>		
BRST	Becchi–Rouet–Stora–Tyutin (cohomology; the homogeneous Slavnov–Taylor identity in the present paper)	—
GR	General Relativity	—
ABJ	Adler–Bell–Jackiw (the 1969 chiral anomaly)	—
AHSS	Atiyah–Hirzebruch spectral sequence	—
APS	Atiyah–Patodi–Singer (index theorem; eta invariant)	—
IIB	Type IIB string theory	—
SUGRA	Supergravity	—
TQFT	Topological Quantum Field Theory	—
WGC	Weak Gravity Conjecture	—
QCD	Quantum Chromodynamics	—
QED	Quantum Electrodynamics	—
QFT	Quantum Field Theory	—
QM	Quantum Mechanics	—
WCS	Wu–Chern–Simons (differential cohomology; the framework for the global completion theorem of BI-6)	—

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Symbol	Meaning	Primary link
GS	Green–Schwarz (anomaly-cancellation mechanism via tensor inflow)	—
<i>Standard Model and beyond</i>		
EFT	Effective Field Theory	—
EW	Electroweak	—
GUT	Grand Unified Theory	—
SM	Standard Model of particle physics	—
SSB	Spontaneous Symmetry Breaking (closed-system; generalized by GSSB)	—
SUSY	Supersymmetry	—
<i>Higher-dimensional constructions</i>		
AdS	Anti-de Sitter spacetime	—
AdS/CFT	Anti-de Sitter / Conformal Field Theory correspondence	—
CFT	Conformal Field Theory	—
KK	Kaluza–Klein (mode expansion / dimensional reduction)	—
RS	Randall–Sundrum (warped extra-dimension model)	—
<i>Quantum gravity and cosmology</i>		
Λ CDM	Standard cosmological model with cosmological constant + cold dark matter	—
LQG	Loop Quantum Gravity	—
TOV	Tolman–Oppenheimer–Volkoff (equation)	—
<i>Foundations of quantum mechanics</i>		
CHSH	Clauser–Horne–Shimony–Holt (inequality)	—
EPR	Einstein–Podolsky–Rosen (paradox / argument)	—
<i>Open-system / extremal-principle sub-acronyms</i>		
FEP	Free Energy Principle (boundary inference avatar of UEP)	GG-A7-Qi4, Qi-EP
MEPP	Maximum Entropy Production Principle (heuristic boundary avatar of UEP)	GG-A7-Qi4, Qi-EP
PLA	Principle of Least Action (closed/bulk avatar of UEP; standard variational dynamics)	GG-A7-Qi4, Qi-EP
PLT	Principle of Least Time (operational/boundary avatar of UEP; fastest lawful path)	GG-A7-Qi4, Qi-EP

Figure 8: hierarchy of program-defined acronyms in the GG-Theory program. Top: the framework name (GG-Theory) and its constitutional charter (GG-Constitution). Tier 2: the four axioms (AX-1–AX-4) and seven postulates (POS-1–POS-7) that together define the admissibility predicate. Tier 3: three bulk objects forced by the constitution — GG-6 (6D bulk), BI-6 (modified Bianchi identity; the principal object of the present paper), OGN-5 (integer ledger). Tier 4: the boundary law GDOS that the bulk projects to via OGGEp, with input acronym SIP. Tier 5: the two four-pillar structures, Gi-4 (geometric flank, with sub-acronyms GEP/CEP blocks of GFF and DGI/GSSB/GA stages of GRIP) and Qi-4 (quantum-information flank, with Qi-EP sub-acronyms

PLT/MEPP/FEP, Qi-CON sub-concepts Conformal/Lévy/Fractal/1/ f , Qi-HAL sub-acronyms MeP-MoS/NHT/HAL/BFFT, the boundary completion $\text{OH}+\text{IH}=\text{EIC}$, and the runtime carrier SeOS (Selfish Open System) underneath). Bottom: the bioenergetic Arisaka constant $\mathbf{Q}_B$ (derived secondary quantity), with the Arisaka equation $J_E/\mathbf{Q}_B = J_e = J_B = J_{Q_i}$ that locks the four boundary currents through one conversion factor.

With the dictionary in hand, the course proper begins. Read the next appendix slowly: the descent chain it builds is the one tool that every later appendix reuses without modification.

Global naming convention for cross-cohort citation

Beginning with [Arisaka \(2026, A2-GG6\)](#) and extended across the cohort, each load-bearing item of the GG-Theory program carries a globally unique identifier of the form *Prefix-Suffix-N*, where the Prefix names the item type and the Suffix names the paper:

- **T-Suffix-N** for the N th Theorem. The GG-A3-BI6 global identifiers were assigned under an earlier sectioning of this paper and no longer track its printed section numbers — T-BI6-9.1, for instance, is the trace-ratio lock, which prints as Theorem 8.1. They are kept fixed rather than regenerated, so that pointers already carried by companion papers continue to resolve, and the concordance in Appendix J maps every one of them to the theorem number it prints as. A global identifier is unique; it is not a section number.
- **L-Suffix-N** for the N th Lemma.
- **C-Suffix-N** for the N th Corollary.
- **D-Suffix-N** for the N th Definition.
- **R-Suffix-N** for the N th Remark.
- **P-Suffix-N** for the N th Prediction.
- **O-Suffix-N** for the N th Objection (e.g., O-BI6-1 is the first Common Objection of the present paper).
- **F-Suffix-N** for the N th Falsifier (e.g., F-BI6-1 is absolute proton stability).
- **I-Suffix-N** for the N th Innovation (e.g., I-BI6-6 is the OGN-5-ledger-as-theorem innovation above).
- **PS-Suffix-N** for the N th Problem-Solved entry (§17).

The canonical Suffix list for the twenty-two cohort papers is: *Constitution* (A1), *GG6* (A2), *BI6* (A3, this paper), *Euclid* (A4), *GDOS* (A5), *GRIP* (A6), *Qi4* (A7), *QM* (A8), *SM* (A9), *Cosmos* (A10), *Origin* (A11), together with the phenomenology preprints *KK* (P1), *N3* (P2), *Electron* (P3), *Proton* (P6), the foundations preprints *Real* (P4) and *CPT* (P5), and the consequence preprints *Kaxion* (C1), *Hubble* (C2), *BH* (C3), and the white papers *Symmetry* (W1) and *Holonomy* (W2). The four axioms (AX-1...AX-4) and seven postulates (POS-1...POS-7) of [Arisaka \(2026, A1-Constitution\)](#) keep their existing cohort-wide names; the global identifier is the same item viewed from outside its home paper.

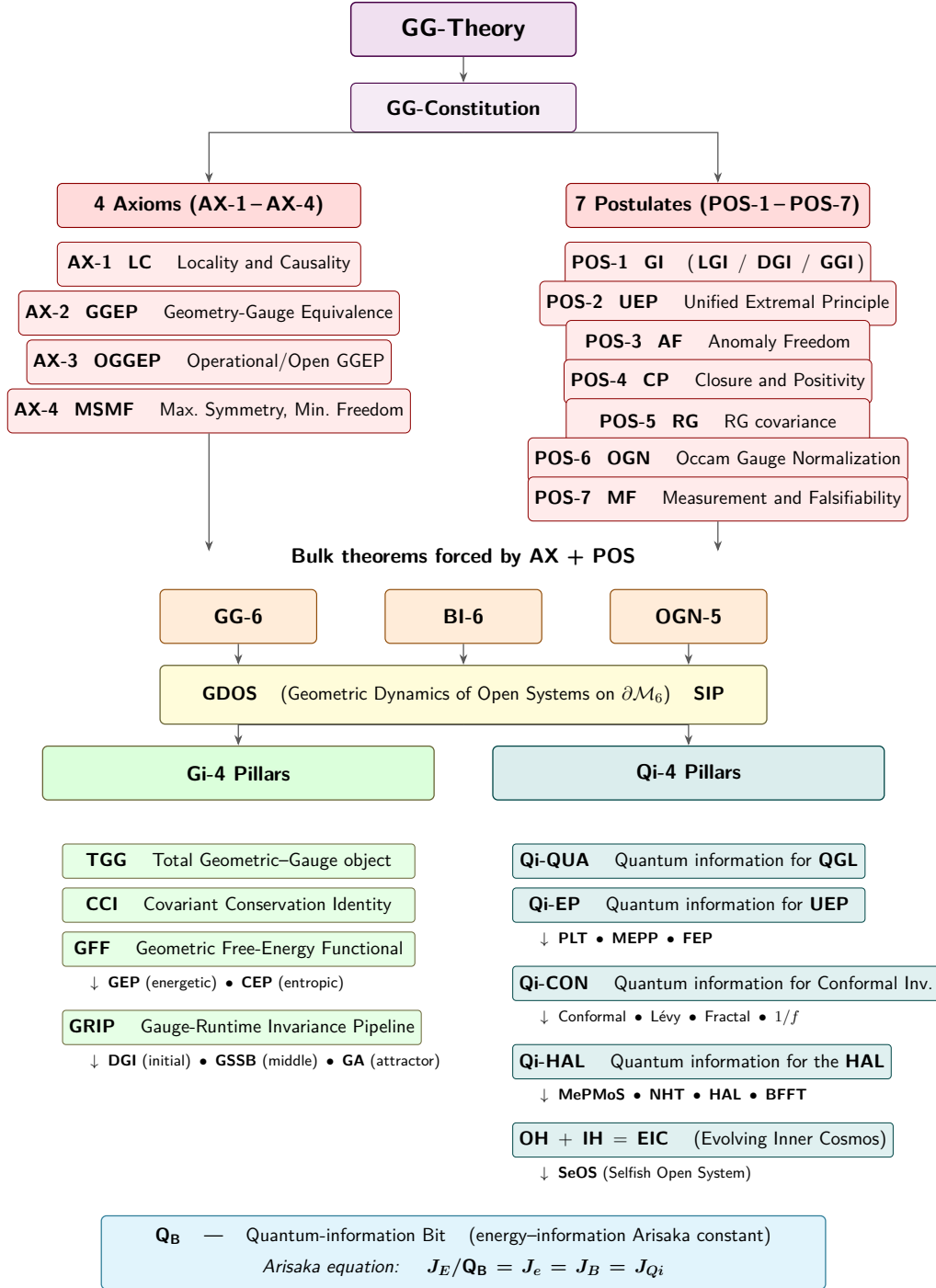


Figure 8: How the program’s own names are related to each other. The chart is a hierarchy, not a list: the charter sits at the top, the geometry it forces below it, the boundary law below that, and the derived pillars at the foot, so the position of a name on the chart tells the reader what it depends on. Names at the same level are siblings introduced by the same construction. The purpose is orientation rather than definition — the meanings are in the abbreviations table — and a reader who has lost track of which name belongs to which layer can recover it here in one glance.

B Complete descent proof and BRST cancellation

This appendix expands the structural proof of Section 5 with every intermediate step displayed. No computation is left implicit.

What this appendix teaches. The descent formalism is the single most reusable tool in this paper: the same three-step chain — transgress a characteristic form to a Chern–Simons potential, vary it to obtain the descent two-form, cancel the one-loop variation with a counterterm — runs identically in four, six, and ten dimensions, and a student who masters it here owns the corresponding chapter of every string-theory textbook. The standard treatments are Bilal (2008) and Harvey (2005) (lecture notes, closest in level to this appendix), Weinberg (1996) (Vol. II, Ch. 22), Bertlmann (1996) (the cohomological route taken here), and Fujikawa and Suzuki (2004) (the path-integral route).

Physics intuition: where the descent comes from. The one-loop anomaly is, at bottom, Fujikawa’s observation that the fermion measure is not invariant: under a gauge variation the measure shifts by the index density of the Dirac operator — a characteristic form. The Wess–Zumino consistency condition then says that whatever the anomaly is, it must be the gauge variation of *something*; the descent chain is the systematic construction of that something. Nothing in the chain is specific to the spectrum or the dimension: it is bookkeeping for the single statement “the anomaly is a characteristic class, opened once by a transgression and once by a variation.” That is why the identical chain serves the ten-dimensional mechanism of Appendix C and the six-dimensional mechanism proved here.

B.1 Chern–Simons three-forms for each sector

The BI-6 four-form involves four curvature sectors. Each sector has a Chern–Simons three-form potential.

Gravitational sector. For the spin connection ω with curvature $R = d\omega + \omega \wedge \omega$, the Lorentz Chern–Simons three-form is

$$\omega_{3L} = \text{tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right). \quad (124)$$

It satisfies

$$d\omega_{3L} = \text{tr}(R \wedge R) = \text{tr} R^2. \quad (125)$$

Under an infinitesimal local Lorentz variation with parameter Θ :

$$\delta_{\Theta} \omega_{3L} = d(\text{tr}(\Theta d\omega)). \quad (126)$$

In the descent notation, the two-form

$$\omega_{2L}^{(1)} = \text{tr}(\Theta d\omega) \quad (127)$$

satisfies

$$\delta_{\Theta} \omega_{3L} - d\omega_{2L}^{(1)} = 0. \quad (128)$$

Non-Abelian SU(5) sector. For the SU(5) connection A_5 with curvature $F_5 = dA_5 + A_5 \wedge A_5$:

$$\omega_{3,5} = \text{Tr} \left(A_5 \wedge dA_5 + \frac{2}{3} A_5 \wedge A_5 \wedge A_5 \right). \quad (129)$$

It satisfies

$$d\omega_{3,5} = \text{Tr}(F_5 \wedge F_5) = \text{Tr} F_5^2. \quad (130)$$

Under an infinitesimal gauge variation with parameter Λ_5 :

$$\omega_{2,5}^{(1)} = \text{Tr}(\Lambda_5 dA_5) \quad (131)$$

satisfies

$$\delta_{\Lambda_5} \omega_{3,5} - d\omega_{2,5}^{(1)} = 0. \quad (132)$$

Abelian U(1)_B sector. For the Abelian connection A_B with $F_B = dA_B$:

$$\omega_{3B} = A_B \wedge F_B = A_B \wedge dA_B. \quad (133)$$

It satisfies

$$d\omega_{3B} = F_B \wedge F_B = F_B^2. \quad (134)$$

Under an infinitesimal gauge variation $\delta A_B = d\lambda_B$:

$$\delta\omega_{3B} = d\lambda_B \wedge F_B + A_B \wedge d(d\lambda_B) = d\lambda_B \wedge F_B = d(\lambda_B F_B), \quad (135)$$

so the two-form is

$$\omega_{2B}^{(1)} = \lambda_B F_B, \quad \delta\omega_{3B} = d\omega_{2B}^{(1)}. \quad (136)$$

Abelian U(1)_{Qi} sector. Identically,

$$\omega_{3,Qi} = A_{Qi} \wedge F_{Qi}, \quad d\omega_{3,Qi} = F_{Qi}^2, \quad \omega_{2,Qi}^{(1)} = \lambda_{Qi} F_{Qi}. \quad (137)$$

B.2 The composite BI-6 Chern–Simons three-form

Assembling all four sectors with the BI-6 coefficients $(1, -1, -2/5, -2/5)$:

$$\boxed{\omega_3^{\text{BI6}} = \omega_{3L} - \omega_{3,5} - \frac{2}{5} \omega_{3B} - \frac{2}{5} \omega_{3,Qi}.} \quad (138)$$

Verification:

$$\begin{aligned}
d\omega_3^{\text{BI6}} &= d\omega_{3L} - d\omega_{3,5} - \frac{2}{5} d\omega_{3B} - \frac{2}{5} d\omega_{3,Qi} \\
&= \text{tr } R^2 - \text{Tr } F_5^2 - \frac{2}{5} F_B^2 - \frac{2}{5} F_{Qi}^2 \\
&= X_4^{\text{BI6}}. \quad \checkmark
\end{aligned} \tag{139}$$

The composite two-form is

$$\boxed{\omega_2^{\text{BI6},(1)} = \omega_{2L}^{(1)} - \omega_{2,5}^{(1)} - \frac{2}{5} \omega_{2B}^{(1)} - \frac{2}{5} \omega_{2,Qi}^{(1)}} \tag{140}$$

Explicitly, substituting (127), (131), (136), and (137):

$$\omega_2^{\text{BI6},(1)} = \text{tr}(\Theta d\omega) - \text{Tr}(\Lambda_5 dA_5) - \frac{2}{5} \lambda_B F_B - \frac{2}{5} \lambda_{Qi} F_{Qi}. \tag{141}$$

Verification of the descent relation $\delta\omega_3^{\text{BI6}} = d\omega_2^{\text{BI6},(1)}$:

$$\begin{aligned}
\delta\omega_3^{\text{BI6}} - d\omega_2^{\text{BI6},(1)} &= \left(\delta\omega_{3L} - d\omega_{2L}^{(1)} \right) - \left(\delta\omega_{3,5} - d\omega_{2,5}^{(1)} \right) \\
&\quad - \frac{2}{5} \left(\delta\omega_{3B} - d\omega_{2B}^{(1)} \right) - \frac{2}{5} \left(\delta\omega_{3,Qi} - d\omega_{2,Qi}^{(1)} \right) \\
&= 0 - 0 - 0 - 0 = 0. \quad \checkmark
\end{aligned} \tag{142}$$

B.3 The improved field strength H_3 and its gauge invariance

Define

$$H_3 = dB_2 + \omega_3^{\text{BI6}}. \tag{143}$$

Then

$$dH_3 = d\omega_3^{\text{BI6}} = X_4^{\text{BI6}}. \tag{144}$$

Assign the transformation law

$$\boxed{\delta B_2 = -\omega_2^{\text{BI6},(1)} + d\Lambda_1}, \tag{145}$$

where Λ_1 is a one-form gauge parameter of B_2 . Then:

$$\begin{aligned}
\delta H_3 &= d(\delta B_2) + \delta\omega_3^{\text{BI6}} \\
&= d(-\omega_2^{\text{BI6},(1)} + d\Lambda_1) + \delta\omega_3^{\text{BI6}} \\
&= -d\omega_2^{\text{BI6},(1)} + d\omega_2^{\text{BI6},(1)} \\
&= 0. \quad \checkmark
\end{aligned} \tag{146}$$

The improved field strength is gauge invariant.

Physics intuition. Equation (145) is the heart of the mechanism, and it repays a moment of contemplation. The two-form B_2 is declared to shift under gauge and Lorentz transformations — a classical transformation law, chosen once. The prize is twofold: the *field strength* H_3 becomes exactly invariant (so the kinetic term is healthy), while the *action* term $\int B_2 \wedge X_4$ becomes exactly anomalous — with the opposite sign to the fermion loop. The quantum sin of the chiral fermions is paid for by a classical tithe from the tensor sector. In the ten-dimensional mechanism the identical trade is made with $\int B_2 \wedge X_8$ (Appendix C).

B.4 The descent chain: seven-form, six-form, and one-loop variation

Define the seven-form

$$I_7^{(0)} = \frac{1}{2} \omega_3^{\text{BI6}} \wedge X_4^{\text{BI6}}. \quad (147)$$

Verify:

$$\begin{aligned} dI_7^{(0)} &= \frac{1}{2} d\omega_3^{\text{BI6}} \wedge X_4^{\text{BI6}} + \frac{1}{2} \omega_3^{\text{BI6}} \wedge dX_4^{\text{BI6}} \\ &= \frac{1}{2} X_4^{\text{BI6}} \wedge X_4^{\text{BI6}} + 0 \\ &= I_8^{\text{BI6}}. \quad \checkmark \end{aligned} \quad (148)$$

(The second term vanishes because $dX_4^{\text{BI6}} = 0$, since X_4 is built from characteristic classes.)

Take the gauge/Lorentz variation of $I_7^{(0)}$:

$$\begin{aligned} \delta I_7^{(0)} &= \frac{1}{2} \delta \omega_3^{\text{BI6}} \wedge X_4^{\text{BI6}} + \frac{1}{2} \omega_3^{\text{BI6}} \wedge \delta X_4^{\text{BI6}} \\ &= \frac{1}{2} \delta \omega_3^{\text{BI6}} \wedge X_4^{\text{BI6}} \quad (\text{since } \delta X_4^{\text{BI6}} = 0) \\ &= \frac{1}{2} d\omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}} \quad (\text{from (142)}) \\ &= d\left(\frac{1}{2} \omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}}\right) \quad (\text{since } dX_4 = 0). \end{aligned} \quad (149)$$

Therefore the consistent anomaly six-form is

$$\boxed{I_6^{(1)} = \frac{1}{2} \omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}}}, \quad (150)$$

satisfying the descent equation

$$\delta I_7^{(0)} = dI_6^{(1)}. \quad (151)$$

The one-loop effective action has the anomalous variation

$$\delta \Gamma_{\text{1-loop}} = \kappa \int_{M_6} I_6^{(1)} = \frac{\kappa}{2} \int_{M_6} \omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}}, \quad (152)$$

where κ is a normalization constant (often $\kappa = 2\pi i$).

B.5 Green–Schwarz counterterm and exact cancellation

The Green–Schwarz counterterm is

$$S_{\text{GS}} = \frac{\kappa}{2} \int_{M_6} B_2 \wedge X_4^{\text{BI6}}. \quad (153)$$

Its variation under gauge/Lorentz transformations:

$$\begin{aligned} \delta S_{\text{GS}} &= \frac{\kappa}{2} \int_{M_6} \delta B_2 \wedge X_4^{\text{BI6}} + \frac{\kappa}{2} \int_{M_6} B_2 \wedge \delta X_4^{\text{BI6}} \\ &= \frac{\kappa}{2} \int_{M_6} (-\omega_2^{\text{BI6},(1)} + d\Lambda_1) \wedge X_4^{\text{BI6}} + 0 \\ &= -\frac{\kappa}{2} \int_{M_6} \omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}} + \frac{\kappa}{2} \int_{M_6} d\Lambda_1 \wedge X_4^{\text{BI6}} \\ &= -\frac{\kappa}{2} \int_{M_6} \omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}} + \frac{\kappa}{2} \int_{M_6} d(\Lambda_1 \wedge X_4^{\text{BI6}}) \\ &= -\frac{\kappa}{2} \int_{M_6} \omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}} + \frac{\kappa}{2} \int_{\partial M_6} \Lambda_1 \wedge X_4^{\text{BI6}}. \end{aligned} \quad (154)$$

On a closed six-manifold (or in the bulk away from boundaries), the boundary integral vanishes:

$$\delta S_{\text{GS}}|_{\text{bulk}} = -\frac{\kappa}{2} \int_{M_6} \omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}}. \quad (155)$$

Comparing (152) and (155):

$$\begin{aligned} \delta \Gamma_{1\text{-loop}} + \delta S_{\text{GS}} &= \frac{\kappa}{2} \int_{M_6} \omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}} - \frac{\kappa}{2} \int_{M_6} \omega_2^{\text{BI6},(1)} \wedge X_4^{\text{BI6}} \\ &= 0. \end{aligned} \quad (156)$$

$$\boxed{\delta(\Gamma_{1\text{-loop}} + S_{\text{GS}}) = 0.} \quad (157)$$

This completes the descent proof.

Physics intuition. The cancellation is exact, not approximate, because both sides of the ledger are descendants of the same characteristic class: the fermion loop produces $+\frac{\kappa}{2} \int \omega_2^{(1)} \wedge X_4$ by descent from $I_8 = \frac{1}{2} X_4^2$, and the counterterm produces the same object with a minus sign because δB_2 was *defined* to be the descent two-form. There is no fine-tuning to preserve under radiative corrections: the statement is cohomological, and it holds at the level of the effective action to all orders in the background fields. A student comparing with the ten-dimensional case (Appendix C) will find the same proof with X_8 in place of the second X_4 — the six-dimensional perfect square is the special case in which the two factors coincide, which is exactly why the rank-one criterion of Appendix D has no ten-dimensional analogue.

B.6 BRST formulation and Wess–Zumino consistency

In BRST language, replace $\delta \rightarrow s$ (the nilpotent BRST operator). The proof becomes:

$$sX_4^{\text{BI6}} = 0, \quad (158)$$

$$s\omega_3^{\text{BI6}} - d\omega_2^{\text{BI6},(1)} = 0, \quad (159)$$

$$sB_2 = -\omega_2^{\text{BI6},(1)}, \quad (160)$$

$$\begin{aligned} s(H_3) &= s(dB_2 + \omega_3^{\text{BI6}}) = d(sB_2) + s\omega_3^{\text{BI6}} \\ &= -d\omega_2^{\text{BI6},(1)} + d\omega_2^{\text{BI6},(1)} = 0, \end{aligned} \quad (161)$$

$$s\Gamma_{1\text{-loop}} = \frac{\kappa}{2} \int \omega_2^{\text{BI6},(1)} \wedge X_4, \quad (162)$$

$$sS_{\text{GS}} = -\frac{\kappa}{2} \int \omega_2^{\text{BI6},(1)} \wedge X_4, \quad (163)$$

$$s(\Gamma_{1\text{-loop}} + S_{\text{GS}}) = 0. \quad (164)$$

Therefore the anomaly cocycle is BRST-exact in the enlarged field space that includes B_2 . The Slavnov–Taylor identity $\mathcal{S}(\Gamma) = 0$ is satisfied, and no ghost-number-one cohomology class obstructs the quantum consistency of the theory.

The Wess–Zumino consistency condition requires

$$\delta_{\epsilon_1} \mathcal{A}(\epsilon_2) - \delta_{\epsilon_2} \mathcal{A}(\epsilon_1) = \mathcal{A}([\epsilon_1, \epsilon_2]). \quad (165)$$

In BRST language this is $s\mathcal{A} = 0$. Since $\mathcal{A} = \frac{\kappa}{2} \int \omega_2^{(1)} \wedge X_4$ and $sX_4 = 0$, one has

$$s\mathcal{A} = \frac{\kappa}{2} \int s\omega_2^{(1)} \wedge X_4 = \frac{\kappa}{2} \int d\omega_1^{(2)} \wedge X_4 = \frac{\kappa}{2} \int d(\omega_1^{(2)} \wedge X_4) = 0 \quad (166)$$

on a closed manifold. The Wess–Zumino condition is satisfied automatically by the descent construction.

The descent engine is dimension-blind: nothing in this appendix cared that six is six. The natural next stop is the arena where the engine first ran — ten-dimensional string theory — both to see the machinery at full historical scale and to learn, by contrast, exactly which of BI-6’s features are inherited and which are new. One intuition to carry across the seam: in ten dimensions the hard work is done by the choice of gauge group; watch for where the six-dimensional story reassigns that work to the choice of spectrum.

C The ten-dimensional Green–Schwarz mechanism worked from scratch, and the six-dimensional dictionary

This appendix is the string-theory half of the course. It derives, from scratch and at graduate-student resolution, the two arithmetic miracles that make the 1984 Green–Schwarz mechanism work in ten dimensions — the gaugino count 496 and the vanishing of the irreducible sixth-order Casimir

at $\text{SO}(32)$ — and then translates every move into the six-dimensional language of this paper. A student who works through this appendix and Appendix B will know exactly what BI-6 inherits from string theory, and exactly where it parts company. The textbook homes for the material are [Green, Schwarz, and Witten \(1987\)](#) (Vol. II, Ch. 13), [Polchinski \(1998b\)](#) (Ch. 12), and [Becker, Becker, and Schwarz \(2007\)](#) (Ch. 5); the original paper is [Green and Schwarz \(1984\)](#).

C.1 The setting: why ten dimensions is a crisis

Type I and heterotic string theory descend at low energy to ten-dimensional $\mathcal{N} = 1$ supergravity coupled to super-Yang–Mills with some gauge group G . The fermion content is chiral: a positive-chirality gravitino ψ_M , a negative-chirality dilatino λ , and positive-chirality gauginos χ^a in the adjoint of G , of dimension $n = \dim G$. In ten dimensions the local anomaly is encoded by a twelve-form I_{12} , and each chiral species contributes through the index densities (the same $\hat{A}$ -genus machinery derived for six dimensions in Appendix H): gravitational terms through $\text{tr } R^2$, $\text{tr } R^4$, $\text{tr } R^6$, gauge terms through $\text{Tr } F^2$, $\text{Tr } F^4$, $\text{Tr } F^6$ (adjoint traces), and mixed terms.

Physics intuition. In four dimensions the anomaly is a box that the Standard Model escapes by a representation-theoretic conspiracy among quarks and leptons. In ten dimensions the box has many more walls — a twelve-form has room for $\text{tr } R^6$, $\text{Tr } F^6$, and a zoo of mixed monomials — and no conspiracy among ordinary multiplets can close them all. The discovery of 1984 was that for precisely two gauge groups the anomaly does not have to vanish identically: it factorizes, and a single two-form field can absorb it.

C.2 Miracle one: the gaugino count 496

The irreducible gravitational term $\text{tr } R^6$ cannot be absorbed by any Green–Schwarz counterterm: a factorized polynomial $X_4 \wedge X_8$ contains only products of lower forms, never the irreducible twelve-form $\text{tr } R^6$. Its coefficient must therefore vanish identically. Summing the index densities of the three chiral species, the gravitino and dilatino contribute a fixed amount, and each gaugino contributes the standard spin-1/2 unit, so the total $\text{tr } R^6$ coefficient is proportional to

$$(n - 496) \text{tr } R^6, \tag{167}$$

where the -496 is the gravitino-plus-dilatino contribution ([Alvarez-Gaumé and Witten, 1984](#), [Green and Schwarz, 1984](#)). Anomaly freedom therefore *forces*

$$\boxed{n = \dim G = 496} \tag{168}$$

Two compact groups have dimension 496 and admit the required trace structure: $\text{SO}(32)$ ($\dim = 32 \cdot 31/2 = 496$) and $E_8 \times E_8$ ($\dim = 2 \times 248 = 496$).

Physics intuition. The gravitational sector fixes the *size* of the gauge sector before any gauge structure is examined: the gravitino’s irreducible anomaly must be paid for by exactly 496 gauginos,

no more, no fewer. This is the ten-dimensional ancestor of the gravitational completion of Appendix H, where the irreducible p_2 (equivalently $\text{tr } R^4$) coefficient of the six-dimensional ledger forces the counts $n_{\text{SD}} = 4\Omega$ and $n_F = 32\Omega$. Same logic, one dimension pair down: an irreducible curvature monomial that no counterterm can touch dictates a spectrum count.

C.3 Miracle two: the trace identities at $\text{SO}(32)$

The gauge sector must next dispose of the irreducible $\text{Tr } F^6$. For $\text{SO}(n)$, the adjoint traces reduce to fundamental traces (Green, Schwarz, and Witten, 1987; the derivation is an exercise in antisymmetric tensor products):

$$\text{Tr } F^2 = (n - 2) \text{tr } F^2, \quad (169)$$

$$\text{Tr } F^4 = (n - 8) \text{tr } F^4 + 3 (\text{tr } F^2)^2, \quad (170)$$

$$\text{Tr } F^6 = (n - 32) \text{tr } F^6 + 15 \text{tr } F^2 \text{tr } F^4. \quad (171)$$

At $n = 32$ — and only there in the $\text{SO}(n)$ family — the irreducible sixth-order Casimir drops out:

$$\boxed{\text{Tr } F^6|_{\text{SO}(32)} = 15 \text{tr } F^2 \text{tr } F^4, \quad \text{Tr } F^2|_{\text{SO}(32)} = 30 \text{tr } F^2.} \quad (172)$$

For E_8 the same disposal is automatic for a different reason: E_8 has *no* independent fourth- or sixth-order Casimirs at all, so $\text{Tr } F^4 = \frac{1}{100}(\text{Tr } F^2)^2$ and $\text{Tr } F^6 = \frac{1}{7200}(\text{Tr } F^2)^3$ identically. With the irreducible terms gone, the surviving reducible twelve-form assembles into the celebrated factorization

$$I_{12} = X_4 \wedge X_8, \quad X_4 = \text{tr } R^2 - \frac{1}{30} \text{Tr } F^2, \quad (173)$$

with X_8 a specific closed eight-form built from $\text{tr } R^4$, $(\text{tr } R^2)^2$, $\text{tr } R^2 \text{Tr } F^2$, $\text{Tr } F^4$, and $(\text{Tr } F^2)^2$, whose exact coefficients the student should verify once in life against Green, Schwarz, and Witten (1987) (Ch. 13) or Polchinski (1998b) (Ch. 12). The factor $1/30$ in X_4 is the $\text{SO}(32)$ trace conversion of (172). The modified Bianchi identity $\text{d}H_3 = X_4$ and the counterterm $\int B_2 \wedge X_8$ then cancel the remainder by the descent argument of Appendix B, applied verbatim.

Physics intuition. Ten-dimensional string theory passes the anomaly examination by *choosing the group*: the arithmetic facts $n - 496 = 0$ and $n - 32 = 0$ (or the Casimir poverty of E_8) are properties of the gauge group itself, and the matter content is then dictated by supersymmetry — one adjoint gaugino, nothing else to decide. Six-dimensional GG-6 passes the same examination by *choosing the spectrum*: the gauge group $\text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$ is fixed constitutionally, and the irreducible quartic Casimir is killed not by a group identity but by the core-block combination of Appendix F,

$$B_C = B_{10} + B_5 + 2B_{\bar{5}} = (-3) + 1 + 2 = 0 \quad (\text{spectrum combination}), \quad (174)$$

the exact structural counterpart of $n - 32 = 0$ (group selection). Hold those two boxed equations side by side and the similarity and the difference of the two programs are both visible at a glance.

C.4 The six-dimensional dictionary

Table 13 translates each move of the ten-dimensional mechanism into its six-dimensional counterpart, with the section of this paper where the counterpart is derived.

Table 13: Every move of the ten-dimensional Green–Schwarz mechanism with its counterpart here. The table is a dictionary rather than a comparison: each row is one step of the original argument and the six-dimensional step that plays the same part. Read it downward and it is a proof sketch of the present theorem in the vocabulary of the familiar one. The rows where the dictionary has no entry are the informative ones, and they are the rows where supersymmetry did the work in ten dimensions and something else does it here.

Move	10D heterotic / Type I	6D BI-6 (this paper)
Anomaly polynomial	I_{12} (twelve-form)	I_8 (eight-form), Sec. 3
Irreducible gravitational term	$\text{tr } R^6$; killed by $n = 496$	p_2 (i.e. $\text{tr } R^4$); killed by the tensor/singlet completion $n_{\text{SD}} = 4\Omega$, $n_F = 32\Omega$ (App. H)
Irreducible gauge term	$\text{Tr } F^6$; killed by $n = 32$ (group identity) or E_8 Casimir poverty	$\text{Tr } F_5^4$; killed by the core-block combination $B_C = 0$ (App. F)
What is chosen	The gauge group ($\text{SO}(32)$ or $E_8 \times E_8$); matter then fixed by SUSY	The spectrum (core blocks, orbits, completion); group fixed by the constitution
Factorization	$I_{12} = X_4 \wedge X_8$ (two independent forms)	$I_8 = \frac{1}{2} X_4^2$ (perfect square; rank one forced, Thm. 5.1)
Coefficient lock	$1/30$ from $\text{Tr } F^2 = 30 \text{tr } F^2$	$2/5$ from $\text{Tr}(B_0^2)/\text{Tr}(Y_0^2) = (L-N)/L$ (Sec. 9)
Tensor field	B_2 with $dH_3 = X_4$; counterterm $\int B_2 \wedge X_8$	B_2 with $dH_3 = X_4$; counterterm $\frac{\kappa}{2} \int B_2 \wedge X_4$ (App. B)
Integer data output	496, 30, 2 groups	$(M, N, L; J, K) = (1, 3, 5; 7, 35)$ (App. E)
Residual freedom	Compactification landscape ($\sim 10^{500}$)	Finite Diophantine family; $\Omega \in 25\mathbb{Z}$ (App. G)
Global completion	Witten-type checks; Monnier–Moore for 6D SUGRA descendants	Intended-class theorem + computed phase (App. I, Lemma I.10)
Textbook home	Green, Schwarz, and Witten (1987) Ch. 13; Polchinski (1998b) Ch. 12; Becker, Becker, and Schwarz (2007) Ch. 5	This paper, Apps. B–I

C.5 What the student should take away

Three lessons, in increasing order of depth. *First*, the descent machinery — Chern–Simons transgression, the Wess–Zumino condition, the counterterm variation — is *identical* in the two settings; learn it once (Appendix B, or Bilal, 2008, Harvey, 2005) and it serves both. *Second*, the two programs close the irreducible terms by opposite strategies: string theory adjusts the *group* until group-theoretic identities kill them; GG-6 fixes the group constitutionally and adjusts the *spectrum* until index sums kill them. Every subsequent divergence — supersymmetry required versus absent, a two-form product versus a perfect square, a landscape versus a Diophantine family — traces back

to that single strategic fork. *Third*, the six-dimensional problem is the more rigid of the two: the perfect-square factorization $I_8 = \frac{1}{2}X_4^2$ admits no analogue of the independent X_8 , so the entire anomaly must collapse onto a single ray (the rank-one criterion), which is why the coefficient vector, and through it the integer ledger, is overdetermined rather than merely constrained. A student who can reproduce (167), (172), the rank-one collapse of Appendix D, and the 2×2 system of Appendix H owns both mechanisms.

Back in six dimensions, the perfect square is the whole game. The product $X_4 \wedge X_8$ of the previous appendix gave string theory two independent dials; the six-dimensional factorization $I_8 = \frac{1}{2}X_4^2$ has one. The rank-one criterion that follows is the precise statement of that rigidity: a symmetric anomaly matrix with ten independent entries must collapse onto a single ray — six 2×2 minors must vanish at once — and this appendix verifies every collapse explicitly.

D Anomaly matrix, rank-one criterion, and coefficient matching

This appendix combines the rank-one eigenvalue computation with the explicit expansion of the anomaly polynomial and all coefficient-matching and vanishing conditions.

Physics intuition: anomaly space as a four-dimensional arena. Collect the four curvature-squared invariants into the vector $Y = (\text{tr } R^2, \text{Tr } F_5^2, F_B^2, F_{Q_i}^2)$; the reducible anomaly is then a quadratic form on this $\mathbb{R}^4$, and a single Green–Schwarz tensor can only ever cancel what lies along a single ray, because it couples to one four-form $X_4 = v \cdot Y$. The rank-one criterion is thus a statement of *dimensional bookkeeping in anomaly space*: one tensor, one ray. Six dimensions is special because the factorized polynomial is a perfect square $\frac{1}{2}(v \cdot Y)^2$, so the microscopic quadratic form must collapse onto that ray exactly — six independent 2×2 minors must vanish. In ten dimensions the corresponding object is a product of two *different* forms, $X_4 \wedge X_8$, and no rank condition of this sharpness arises (Appendix C). This is the precise sense in which the six-dimensional mechanism is more rigid than the ten-dimensional one it descends from.

D.1 Statement

Theorem D.1 (Rank-one criterion, detailed version). *Let the reducible part of a six-dimensional local anomaly polynomial be written in the basis*

$$Y = (\text{tr } R^2, \text{Tr } F_5^2, F_B^2, F_{Q_i}^2)^T \quad (175)$$

as

$$I_8^{\text{red}} = \frac{1}{2}Y^T M Y, \quad (176)$$

where M is a real symmetric 4×4 matrix. Then:

- (a) *A single Green–Schwarz two-form can cancel the reducible anomaly if and only if $\text{rank}(M) = 1$ and M is positive semidefinite.*

- (b) When $\text{rank}(M) = 1$ and $M \succeq 0$, there exists a unique (up to overall sign) unit vector $\hat{v}$ and a positive scalar λ such that $M = \lambda \hat{v} \hat{v}^T$.
- (c) The Green–Schwarz four-form is $X_4 = \sqrt{\lambda} \hat{v} \cdot Y$, and the single tensor cancels the anomaly $I_8 = \frac{1}{2}(\sqrt{\lambda} \hat{v} \cdot Y)^2$.

Proof. **(a) Necessity.** A single tensor couples to exactly one four-form $X_4 = v \cdot Y$ for some vector v . The cancelable anomaly polynomial is therefore

$$I_8^{\text{GS}} = \frac{1}{2}(v \cdot Y)^2 = \frac{1}{2}Y^T(vv^T)Y. \quad (177)$$

The matrix vv^T is an outer product of one vector with itself. Any outer product has rank at most one. Since $v \neq 0$, the rank is exactly one. Therefore $M = vv^T$ has rank one.

(b) Sufficiency. If M has rank one, then M has exactly one nonzero eigenvalue λ , with corresponding eigenvector $\hat{v}$. Positivity of λ is a genuine extra hypothesis and not a consequence of the rank: over $\mathbb{R}$ the matrix $-vv^T$ also has rank one, and it admits no real factorization ww^T . That is precisely the wrong-sign Green–Schwarz case $I_8 = -\frac{1}{2}X_4^2$, which the single-tensor counterterm of Theorem 5.5 cannot cancel. For the BI-6 matrix the sign is settled by explicit computation — Section D.2 gives eigenvalues $\{58/25, 0, 0, 0\}$ — so $\lambda > 0$ is part of the hypothesis of (b), as stated. Then $M = \lambda \hat{v} \hat{v}^T$. Defining $v = \sqrt{\lambda} \hat{v}$ gives $M = vv^T$, so $I_8 = \frac{1}{2}(v \cdot Y)^2$, which is cancelable by one tensor coupling to $X_4 = v \cdot Y$.

(c) Follows immediately from (b). ■

D.2 Explicit computation for BI-6

The BI-6 vector is

$$v = (1, -1, -2/5, -2/5)^T. \quad (178)$$

The outer product is

$$M = vv^T = \begin{pmatrix} 1 \cdot 1 & 1 \cdot (-1) & 1 \cdot (-2/5) & 1 \cdot (-2/5) \\ (-1) \cdot 1 & (-1) \cdot (-1) & (-1) \cdot (-2/5) & (-1) \cdot (-2/5) \\ (-2/5) \cdot 1 & (-2/5) \cdot (-1) & (-2/5) \cdot (-2/5) & (-2/5) \cdot (-2/5) \\ (-2/5) \cdot 1 & (-2/5) \cdot (-1) & (-2/5) \cdot (-2/5) & (-2/5) \cdot (-2/5) \end{pmatrix}. \quad (179)$$

Evaluating each entry:

$$M = \begin{pmatrix} 1 & -1 & -2/5 & -2/5 \\ -1 & 1 & 2/5 & 2/5 \\ -2/5 & 2/5 & 4/25 & 4/25 \\ -2/5 & 2/5 & 4/25 & 4/25 \end{pmatrix}. \quad (180)$$

Eigenvalue computation. For an outer product $M = vv^T$, the nonzero eigenvalue is

$$\lambda = v^T v = |v|^2 = 1^2 + (-1)^2 + (-2/5)^2 + (-2/5)^2 = 1 + 1 + \frac{4}{25} + \frac{4}{25} = 2 + \frac{8}{25} = \frac{58}{25}. \quad (181)$$

The eigenvector corresponding to $\lambda = 58/25$ is v itself. The other three eigenvalues are all zero, with eigenvectors spanning the orthogonal complement of v in $\mathbb{R}^4$.

Rank verification. Since there is exactly one nonzero eigenvalue:

$$\text{rank}(M) = 1. \quad \checkmark \quad (182)$$

Trace and determinant checks.

$$\text{Tr}(M) = 1 + 1 + 4/25 + 4/25 = 58/25 = \lambda. \quad \checkmark \quad (183)$$

$$\det(M) = 0 \quad (\text{rank } 1, \text{ so three zero eigenvalues}). \quad \checkmark \quad (184)$$

The trace equals the single nonzero eigenvalue, and the determinant vanishes, both consistent with rank one.

D.3 The expansion

Let

$$X_4 = \underbrace{\text{tr } R^2}_{Y_1} - \underbrace{\text{Tr } F_5^2}_{Y_2} - \frac{2}{5} \underbrace{F_B^2}_{Y_3} - \frac{2}{5} \underbrace{F_{Q_i}^2}_{Y_4}. \quad (185)$$

Then

$$\begin{aligned} X_4^2 &= (Y_1 - Y_2 - \frac{2}{5}Y_3 - \frac{2}{5}Y_4)^2 \\ &= Y_1^2 + Y_2^2 + \frac{4}{25}Y_3^2 + \frac{4}{25}Y_4^2 \\ &\quad - 2Y_1Y_2 - \frac{4}{5}Y_1Y_3 - \frac{4}{5}Y_1Y_4 \\ &\quad + \frac{4}{5}Y_2Y_3 + \frac{4}{5}Y_2Y_4 + \frac{8}{25}Y_3Y_4. \end{aligned} \quad (186)$$

Therefore

$$\begin{aligned} \frac{1}{2}X_4^2 &= \frac{1}{2}Y_1^2 + \frac{1}{2}Y_2^2 + \frac{2}{25}Y_3^2 + \frac{2}{25}Y_4^2 \\ &\quad - Y_1Y_2 - \frac{2}{5}Y_1Y_3 - \frac{2}{5}Y_1Y_4 \\ &\quad + \frac{2}{5}Y_2Y_3 + \frac{2}{5}Y_2Y_4 + \frac{4}{25}Y_3Y_4. \end{aligned} \quad (187)$$

D.4 The ten coefficient-matching equations

Restoring the curvature notation, the ten required coefficients are:

Table 14: The ten conditions that the coefficients must satisfy for the anomaly polynomial to factorize, each with the monomial it constrains, the value required, and the verification. Every row must hold; a single failure leaves the theory anomalous and therefore inconsistent, not merely inelegant. The fourth column is the check, written out entry by entry, so a reader can repeat any line. Together with the following table this is the whole of the factorization proof in arithmetic form.

Eq.	Monomial	Required value	Verification
(C1)	$(\text{tr } R^2)^2$	1/2	$v_1^2/2 = 1/2 \checkmark$
(C2)	$(\text{Tr } F_5^2)^2$	1/2	$v_2^2/2 = 1/2 \checkmark$
(C3)	$(F_B^2)^2$	2/25	$v_3^2/2 = (4/25)/2 = 2/25 \checkmark$
(C4)	$(F_{Q_i}^2)^2$	2/25	$v_4^2/2 = 2/25 \checkmark$
(C5)	$\text{tr } R^2 \cdot \text{Tr } F_5^2$	-1	$v_1 v_2 = (1)(-1) = -1 \checkmark$
(C6)	$\text{tr } R^2 \cdot F_B^2$	-2/5	$v_1 v_3 = (1)(-2/5) = -2/5 \checkmark$
(C7)	$\text{tr } R^2 \cdot F_{Q_i}^2$	-2/5	$v_1 v_4 = (1)(-2/5) = -2/5 \checkmark$
(C8)	$\text{Tr } F_5^2 \cdot F_B^2$	+2/5	$v_2 v_3 = (-1)(-2/5) = +2/5 \checkmark$
(C9)	$\text{Tr } F_5^2 \cdot F_{Q_i}^2$	+2/5	$v_2 v_4 = (-1)(-2/5) = +2/5 \checkmark$
(C10)	$F_B^2 \cdot F_{Q_i}^2$	4/25	$v_3 v_4 = (-2/5)(-2/5) = 4/25 \checkmark$

All ten equations are satisfied identically because the BI-6 polynomial is a perfect square. This is the content of the rank-one factorization: if the anomaly matrix is $M = vv^T$, then $M_{ij} = v_i v_j$ automatically satisfies all ten conditions with the values listed above.

D.5 The vanishing conditions

The following monomials are *absent* from $\frac{1}{2}X_4^2$ because X_4 contains only squared field strengths, not individual ones:

Table 15: The six monomials that must *not* appear if the anomaly polynomial is to factorize, each with the reason it cannot. These are as load-bearing as the ten conditions of the previous table and are more easily overlooked, because an absent term leaves no trace in a calculation that only checks what is present. The third column gives the structural reason in each case — irreducibility, or the representation content simply not generating the term. A reader auditing the factorization should check this table as carefully as the last.

Eq.	Monomial	Why absent from $\frac{1}{2}X_4^2$
(V1)	$\text{tr } R^4$	Irreducible; cannot be written as a product of 4-forms
(V2)	$\text{Tr } F_5^4 _{\text{irr}}$	Irreducible quartic; not generated by $(\text{Tr } F_5^2)^2$
(V3)	$\text{Tr } F_5^3 F_B$	Odd in F_B ; not in the square of an even polynomial
(V4)	$\text{Tr } F_5^3 F_{Q_i}$	Odd in F_{Q_i}
(V5)	$F_B^3 F_{Q_i}$	Odd mixed Abelian
(V6)	$F_B F_{Q_i}^3$	Odd mixed Abelian

Additionally, the off-diagonal terms $\text{tr } R^2 F_B F_{Q_i}$, $\text{Tr } F_5^2 F_B F_{Q_i}$ are absent because X_4 contains F_B^2 and $F_{Q_i}^2$, but not the individual two-forms F_B and F_{Q_i} .

D.6 Explicit verification of the off-diagonal vanishing conditions

For completeness, the benchmark verification is written explicitly rather than delegated. The required cancellation is elementary once the orbit pairing is spelled out.

For an axial orbit A_n^σ , the two core blocks carry charges $(n/3, 0)$ and $(0, n/3)$. Therefore every monomial containing both F_B and F_{Q_i} vanishes identically on each axial orbit:

$$\sum_{A_n^\sigma} q_B q_{Q_i} = 0, \quad \sum_{A_n^\sigma} q_B^3 q_{Q_i} = 0, \quad \sum_{A_n^\sigma} q_B q_{Q_i}^3 = 0.$$

For a diagonal orbit D_n^σ , the paired core blocks carry charges $(n/3, n/3)$ and $(n/3, -n/3)$. Summing over the pair yields

$$\begin{aligned} \sum_{D_n^\sigma} q_B q_{Q_i} &= \frac{n^2}{9}(1 - 1) = 0, \\ \sum_{D_n^\sigma} q_B^3 q_{Q_i} &= \frac{n^4}{81}(1 - 1) = 0, \quad \sum_{D_n^\sigma} q_B q_{Q_i}^3 = \frac{n^4}{81}(1 - 1) = 0. \end{aligned}$$

The same sign cancellation immediately kills the mixed gravitational and mixed simple-Abelian monomials:

$$\sum_{D_n^\sigma} p_1 q_B q_{Q_i} = 0, \quad \sum_{D_n^\sigma} \text{Tr } F_5^2 q_B q_{Q_i} = 0.$$

The remaining two forbidden cubic monomials, $\text{Tr } F_5^3 F_B$ and $\text{Tr } F_5^3 F_{Q_i}$, vanish already on each core block because the underlying $\text{SU}(5)$ block has vanishing cubic simple-group coefficient $A_{3,C} = 0$. No inter-orbit cancellation is required. The full verification is summarized in Table 16.

Table 16: The eight forbidden terms — the six of Table 15 together with the two off-diagonal monomials — verified explicitly on the benchmark representation content rather than argued from general principles. The second column names the mechanism that removes each term and the third gives the result of the computation. This table exists because the preceding one states reasons and a reader is entitled to the arithmetic: between them, the claim that the anomaly polynomial factorizes is checkable line by line without reconstructing anything.

Term	Mechanism	Result on the benchmark reservoir
$\text{tr } R^4$	Irreducible gravitational term absent from $\frac{1}{2} X_4^2$	absent from any perfect-square factorization; on the reservoir it is killed by the tensor/singlet completion $n_{\text{SD}} = 4\Omega$, $n_F = 32\Omega$ (App. H)
$\text{Tr } F_5^4 _{\text{irr}}$	Irreducible simple-group term absent from $(\text{Tr } F_5^2)^2$	vanishes on each core block by the GG-6 core-block choice
$\text{Tr } F_5^3 F_B$	cubic simple-group coefficient $A_{3,C} = 0$	vanishes blockwise
$\text{Tr } F_5^3 F_{Q_i}$	cubic simple-group coefficient $A_{3,C} = 0$	vanishes blockwise
$F_B^3 F_{Q_i}$	sign symmetry ($q_{Q_i} \rightarrow -q_{Q_i}$) inside D_n^σ	$\sum q_B^3 q_{Q_i} = 0$ orbit by orbit
$F_B F_{Q_i}^3$	sign symmetry ($q_{Q_i} \rightarrow -q_{Q_i}$) inside D_n^σ	$\sum q_B q_{Q_i}^3 = 0$ orbit by orbit
$\text{tr } R^2 F_B F_{Q_i}$	sign symmetry in $q_B q_{Q_i}$	$\sum q_B q_{Q_i} = 0$ orbit by orbit
$\text{Tr } F_5^2 F_B F_{Q_i}$	sign symmetry in $q_B q_{Q_i}$	$\sum q_B q_{Q_i} = 0$ orbit by orbit

The key point is that the benchmark reservoir does not rely on a delicate cancellation among many unrelated species. The off-diagonal vanishings are already built into the orbit architecture itself.

The rank-one collapse fixes the direction of X_4 ; the integers behind that direction are the business of the next appendix. The intuition to hold: the coefficient $-2/5$ is not a fitted number but a trace ratio — a statement about how baryon number sits inside the unified block — and the moment it is recognized as $(L - N)/L$, anomaly freedom stops being analysis and starts dictating arithmetic.

E OGN-5: complete derivation from BI-6

This appendix combines the trace-ratio computation that fixes the BI-6 Abelian coefficient with the full theorem-level derivation of the OGN-5 integer packet. The trace computation (from Arisaka-GG-A2-GG6 (Arisaka, 2026, A2-GG6)) establishes the bridge; the theorem proof closes it.

Physics intuition: integers from consistency, here and in string theory. Every consistent higher-dimensional program ends up minting integers, because the consistency conditions are topological and topology counts. String theory’s mint produces 496 (the gaugino count forced by $\text{tr } R^6$), 30 (the $\text{SO}(32)$ trace conversion), 32 (the D9-brane/orientifold tadpole of Type I, the spacetime face of the same $\text{SO}(32)$), and 24 (the K3 Euler-characteristic tadpole) — each the unique integer solution of an anomaly or tadpole condition (Appendix C; Polchinski, 1998b). The OGN-5 ledger (1, 3, 5; 7, 35) is minted the same way: each integer is the unique solution of an arithmetic condition set by a prior theorem, and the dual-lock identity $K = LJ = M^2 + N^2 + L^2$ is the overdetermination that makes the packet a witness rather than a convention. The difference is what the integers then buy: string theory’s integers fix the gauge group and leave a landscape of compactifications; the GG-6 integers fix the normalizations of a single bulk and leave a finite Diophantine family (Appendix G).

E.1 The trace computation

The primitive hypercharge generator in the fundamental **5** of $\text{SU}(5)$ is

$$Y_0 = \text{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right). \quad (188)$$

Its square is

$$Y_0^2 = \text{diag}\left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{4}, \frac{1}{4}\right). \quad (189)$$

The trace is

$$\text{Tr}(Y_0^2) = 3 \times \frac{1}{9} + 2 \times \frac{1}{4} = \frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}. \quad (190)$$

The primitive baryon generator is

$$B_0 = \text{diag}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right). \quad (191)$$

Its square is

$$B_0^2 = \text{diag}\left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0, 0\right). \quad (192)$$

The trace is

$$\text{Tr}(B_0^2) = 3 \times \frac{1}{9} = \frac{1}{3}. \quad (193)$$

The ratio is

$$\frac{\text{Tr}(B_0^2)}{\text{Tr}(Y_0^2)} = \frac{1/3}{5/6} = \frac{1}{3} \times \frac{6}{5} = \frac{6}{15} = \frac{2}{5}. \quad (194)$$

E.2 General formula for $\text{SU}(L)$ with N colors

For the group $\text{SU}(L)$ with the decomposition $\text{SU}(N) \times \text{SU}(L-N) \times \text{U}(1)_Y$:

$$Y_0 = \text{diag}\left(\underbrace{-\frac{1}{N}, \dots, -\frac{1}{N}}_N, \underbrace{\frac{1}{L-N}, \dots, \frac{1}{L-N}}_{L-N}\right). \quad (195)$$

$$\begin{aligned} \text{Tr}(Y_0^2) &= N \times \frac{1}{N^2} + (L-N) \times \frac{1}{(L-N)^2} \\ &= \frac{1}{N} + \frac{1}{L-N} = \frac{(L-N) + N}{N(L-N)} = \frac{L}{N(L-N)}. \end{aligned} \quad (196)$$

$$B_0 = \text{diag}\left(\underbrace{\frac{1}{N}, \dots, \frac{1}{N}}_N, \underbrace{0, \dots, 0}_{L-N}\right). \quad (197)$$

$$\text{Tr}(B_0^2) = N \times \frac{1}{N^2} = \frac{1}{N}. \quad (198)$$

The ratio:

$$\boxed{\frac{\text{Tr}(B_0^2)}{\text{Tr}(Y_0^2)} = \frac{1/N}{L/[N(L-N)]} = \frac{N(L-N)}{NL} = \frac{L-N}{L}}. \quad (199)$$

E.3 Numerical verification for all small (N, L)

Table 17 evaluates the ratio for every small pair.

Among the small pairs tabulated, the **GG-6** value $2/5$ occurs only at $(N, L) = (3, 5)$, which is the **OGN** pair corresponding to three colors and $\text{SU}(5)$ unification. The full integer family $(L, N) = (5k, 3k)$ realizes $2/5$ for every k ; it is cut down to $k = 1$ by the dual lock together with **AX-4** (Theorem 9.5), not by the table.

Table 17: The general ratio evaluated for every small pair of ledger integers. The first two columns are the pair, the third the exact fraction and the fourth its decimal value. The row in bold is the one the framework selects, and the table is included so that its selection can be seen against the alternatives rather than in isolation: the value is not special-looking, and nothing about it would be suspicious if it were wrong. What makes it a prediction is that the pair is fixed elsewhere, by the cascade, and not chosen to produce this number.

N	L	$(L-N)/L$	Decimal
2	4	1/2	0.500
2	5	3/5	0.600
3	4	1/4	0.250
3	5	2/5	0.400
3	6	1/2	0.500
4	5	1/5	0.200
4	6	1/3	0.333

E.4 The core block as a function of L

The core block $\mathcal{C} = \mathbf{L}(\mathbf{L}-1)/2 \oplus \mathbf{L} \oplus 2\bar{\mathbf{L}} \oplus n_1 \cdot \mathbf{1}$ requires $d_{\mathcal{C}} = (L+1) \cdot A_{\mathcal{C}}$. In the integer normalization used throughout this paper (where $A(\mathbf{5}) = 1$), the indices are

$$A(\mathbf{10}) = 3, \quad A(\mathbf{5}) = 1, \quad A(\bar{\mathbf{5}}) = 1. \quad (200)$$

The core block index is therefore

$$A_{\mathcal{C}} = A(\mathbf{10}) + A(\mathbf{5}) + 2A(\bar{\mathbf{5}}) = 3 + 1 + 2 = 6 = L + 1. \quad (201)$$

The dimension is

$$d_{\mathcal{C}} = \frac{L(L-1)}{2} + 3L + n_1. \quad (202)$$

Setting $d_{\mathcal{C}} = (L+1) \cdot A_{\mathcal{C}} = (L+1)^2$:

$$n_1 = (L+1)^2 - \frac{L(L-1)}{2} - 3L = L^2 + 2L + 1 - \frac{L^2 - L}{2} - 3L = \frac{L^2 - L + 2}{2}. \quad (203)$$

For $L = 5$: $d_{\mathcal{C}} = 10 + 5 + 10 + 11 = 36 = 6^2$, $n_1 = (25 - 5 + 2)/2 = 11$. ✓

E.5 Precise meanings and the main OGN-5 theorem

The cleaned-up meanings are as follows.

Definition E.1 (Meaning of $M, N, L; J, K$). In the intended GG-6 category:

- (i) M is the **primitive tensor/gravitational slot count**. It counts the minimal GS/tensor channel carried by the single BI-6 tensor lattice.
- (ii) N is the **color triplicity index**, written N_c wherever the physical color count rather than the ledger slot is meant. It is the size of the color block carried inside the defining

representation of the unified group. It is *not* the number of fermion generations: that quantity is written N_g , it is not a member of this ledger, and it is fixed by an argument of an entirely different kind (Arisaka, 2026, P2-N3). That $N_c = N_g = 3$ is a coincidence on the boundary, and this paper never uses it.

- (iii) L is the **defining unified block size**. In GG-6 it is the defining dimension of the minimal simple unified group $SU(5)$.
- (iv) J is the **minimal symmetric closure cardinality**. If the largest allowed absolute Abelian orbit level is N , then OGN closure plus sign symmetry require the packet

$$\mathcal{P}_N := \{0, \pm 1, \pm 2, \dots, \pm N\}.$$

The integer J is the cardinality of this packet.

- (v) K is the **total lock index**. It is simultaneously the rectangular product LJ and the quadratic norm of the normalization packet.

E.6 The trace-ratio theorem that fixes the BI-6 Abelian fraction

The cleanest route begins with a general $SU(L)$ block split into a color block of size N and a complementary block of size $L - N$.

Define the two diagonal directions

$$B_{N,L} = \text{diag} \left(\underbrace{\frac{1}{N}, \dots, \frac{1}{N}}_N, \underbrace{0, \dots, 0}_{L-N} \right),$$

$$Y_{N,L} = \text{diag} \left(\underbrace{-\frac{1}{N}, \dots, -\frac{1}{N}}_N, \underbrace{\frac{1}{L-N}, \dots, \frac{1}{L-N}}_{L-N} \right).$$

The first is the baryonic block direction and the second is the traceless hypercharge-like splitting direction. A direct calculation gives

$$\text{Tr}(B_{N,L}^2) = N \cdot \frac{1}{N^2} = \frac{1}{N},$$

$$\text{Tr}(Y_{N,L}^2) = N \cdot \frac{1}{N^2} + (L - N) \cdot \frac{1}{(L - N)^2} = \frac{1}{N} + \frac{1}{L - N} = \frac{L}{N(L - N)}.$$

Therefore

$$\frac{\text{Tr}(B_{N,L}^2)}{\text{Tr}(Y_{N,L}^2)} = \frac{L - N}{L}. \quad (204)$$

Equation (204) is the structural theorem needed by OGN. It is not specific to $L = 5$; it is the

general block formula. In GG-6, BI-6 tells us that the Abelian coefficient must be exactly

$$\frac{2}{5}.$$

Hence OGN closure demands

$$\frac{L - N}{L} = \frac{2}{5}. \quad (205)$$

The main OGN-5 theorem.

Theorem E.2 (OGN-5 theorem). *Within the intended GG-6 category, BI-6 and the unified block structure force uniquely*

$$(M, N, L; J, K) = (1, 3, 5; 7, 35), \quad K = LJ = M^2 + N^2 + L^2.$$

Proof. The proof proceeds in five steps.

Step 1: $M = 1$. GG-6 uses a single primitive Green–Schwarz tensor channel. Equivalently, the tensor lattice is rank one. Therefore the primitive tensor/gravitational slot count is

$$M = 1.$$

Step 2: $L = 5$. By Theorem 9.2, single-tensor Green–Schwarz consistency requires the irreducible quartic and cubic indices of the charged core package to vanish, and both equal $L - 5$; with AX-4 (MSMF) selecting the minimal chiral package,

$$L = 5.$$

This step does not appeal to the smallest-simple-group clause and does not consume the observed low-energy gauge content: it is free of N throughout, which is what makes Step 3 available in the form given below.

Step 3: $N = N_c = 3$. By Theorem 9.4, the dual lock together with the closure-index clause $J = 2N + 1$ gives $N = L \pm \sqrt{L - M^2}$; at $M = 1$ and the $L = 5$ of Step 2 the roots are 3 and 7, and the block embedding $N < L$ retains

$$N = N_c = 3.$$

The trace ratio is not used here. It is computed on the block split of the defining representation, so it would consume the split it concludes; its role in this paper is the independent check of Eq. (108), not a derivation. Nothing in this step bears on the generation count N_g .

Step 4: $J = 2N + 1$. BI-6 does not use an arbitrary closure count. The diagonal orbit system already forces sign symmetry: if level $+n$ occurs, the paired orbit contains level $-n$ as well. OGN

minimality further requires a neutral reference slot 0. Therefore the minimal closed charge packet is

$$\mathcal{P}_N = \{0, \pm 1, \dots, \pm N\}, \quad |\mathcal{P}_N| = 2N + 1.$$

With $N = 3$, this gives

$$J = |\mathcal{P}_N| = 2 \cdot 3 + 1 = 7.$$

Step 5: $K = LJ$ and the quadratic witness identity. Rectangular closure gives

$$K = LJ = 5 \cdot 7 = 35.$$

Then the quadratic norm of the OGN packet is

$$M^2 + N^2 + L^2 = 1^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35 = K.$$

Hence

$$K = LJ = M^2 + N^2 + L^2 = 35.$$

This proves the claim. ■

E.7 Uniqueness and physical interpretation

The previous proof is already sufficient, but there is an instructive uniqueness corollary.

Corollary E.3 (Why the false branch $L = 2$ is excluded). *Assume only $M = 1$, $N = 3$, the minimal symmetric closure cardinality $J = 2N + 1$, and the two closure rules*

$$K = LJ, \quad K = M^2 + N^2 + L^2.$$

Then L solves

$$L^2 - 7L + 10 = 0,$$

with roots $L = 2$ and $L = 5$. The branch $L = 2$ is physically inadmissible, so $L = 5$ is unique.

Proof. Substitute $M = 1$, $N = 3$, $J = 7$:

$$7L = 1 + 9 + L^2 \implies L^2 - 7L + 10 = 0.$$

The roots are $L = 2$ and $L = 5$. The minimal exclusion criterion is already sufficient: $L = 2 < N = 3$, so a color block of size three cannot be embedded inside the unified simple block. This corollary is retained as a consistency check on the closure identities and not as the derivation of L : it takes N as given, and therefore cannot be used to support any statement about N . The derivation of L is Theorem 9.2, which is free of N throughout. The further failures to realize the $SU(5)$ defining stage and the exact BI-6 denominator-5 trace structure are corollaries of this basic obstruction. Therefore only $L = 5$ survives. ■

Physical interpretation. The theorem above gives a precise and cleaner interpretation of the integers.

To summarize the OGN-5 content: $M = 1$ counts the primitive tensor slot, $N = 3$ is the color triplicity index fixed by the BI-6 trace ratio, $L = 5$ is the defining unified block size of $SU(5)$, $J = 7$ is the minimal symmetric closure cardinality of the charge packet $\{0, \pm 1, \pm 2, \pm 3\}$, and $K = 35$ is the total lock index. The reappearance of the numeral 3 as the generation count on the boundary is a downstream consequence, not the primitive definition of N .

The ledger says which integers the spectrum must realize; the core block is the brick engineered to realize them. The intuition: the block pre-pays the two irreducible debts — the cubic and quartic Casimirs, the sins no Green–Schwarz tensor can absorb — internally, within its own thirty-six states, so that assembling blocks into orbits only ever has to balance cancellable coin.

F Core block anomaly polynomial

This appendix computes the anomaly polynomial of one core block $\mathcal{C}^\sigma(q_B, q_{Q_i})$ as defined in Section 7. The key results are that the irreducible quartic index $B_{\mathcal{C}}$ and the cubic index $A_{3,\mathcal{C}}$ both vanish, confirming Proposition 7.5.

A primer on the group-theory indices (for the graduate reader). Four numbers characterize how a representation ρ of $SU(N)$ feeds the anomaly polynomial, and all four are defined by reducing traces in ρ to traces in the fundamental:

$$\mathrm{tr}_\rho F^2 = A_\rho \mathrm{tr} F^2 \quad (\text{quadratic index; normalized to } A(\mathbf{N}) = 1), \quad (206)$$

$$\mathrm{tr}_\rho F^3 = A_3(\rho) \mathrm{tr} F^3 \quad (\text{cubic anomaly coefficient; } A_3(\bar{\rho}) = -A_3(\rho)), \quad (207)$$

$$\mathrm{tr}_\rho F^4 = B_\rho \mathrm{Tr} F^4 + C_\rho (\mathrm{Tr} F^2)^2 \quad (\text{quartic decomposition into irreducible and reducible parts}). \quad (208)$$

The dimension d_ρ completes the set. For the representations a model-builder actually uses, the closed forms are classical (they follow from character expansions of tensor products; see [Slansky, 1981](#) for tables and [Georgi, 1999](#) for the derivations): the fundamental has $(A, A_3, B, C) = (1, 1, 1, 0)$; the two-index antisymmetric $\mathbf{N}(\mathbf{N}-1)/2$ has $(N-2, N-4, N-8, 3)$; the adjoint has $(2N, 0, 2N, 6)$; and every conjugate flips the sign of A_3 while preserving A, B, C . Setting $N = 5$ reproduces every entry of Table 18 below; the reader is encouraged to verify $B_{\mathbf{10}} = 5 - 8 = -3$ once by hand, since that single integer is what the core-block combination $(-3) + 1 + 2 = 0$ must cancel — the six-dimensional counterpart of the $SO(32)$ identity $n - 32 = 0$ of Appendix C.

Physics intuition. A measures how strongly the representation couples to the reducible (Green–Schwarz–cancellable) part of the anomaly; A_3 and B measure the two irreducible sins — the cubic and quartic Casimirs — that no tensor can absorb and that must therefore vanish by *combination*; C is the representation’s own contribution to the reducible quartic. The core block is engineered so that the sins cancel within each block ($A_{3,\mathcal{C}} = B_{\mathcal{C}} = 0$) while the virtues add ($A_{\mathcal{C}} = 6, C_{\mathcal{C}} = 3$):

pre-payment of the irreducible debts, block by block, so that the inter-block Diophantine system of Appendix G needs to manage only the cancellable remainder.

F.1 SU(5) group theory data

Table 18: The group-theoretic data every calculation in this paper draws on: dimensions, indices and anomaly coefficients for the representations that appear. It is gathered here so that no derivation has to restate it and so that a reader checking an anomaly coefficient has one place to verify it. All values are standard and are quoted, not derived; an error in this table would propagate into every result in the paper, which is why it is printed in full rather than cited.

ρ	d_ρ	A_ρ	$A_3(\rho)$	B_ρ	C_ρ
5	5	1	+1	1	0
$\bar{5}$	5	1	-1	1	0
10	10	3	+1	-3	3
$\bar{10}$	10	3	-1	-3	3
24	24	10	0	10	6
1	1	0	0	0	0

The quartic decomposition is

$$\text{tr}_\rho(F_5^4) = B_\rho \text{Tr } F_5^4 + C_\rho (\text{Tr } F_5^2)^2, \quad (209)$$

where Tr denotes the trace in the fundamental. For SU(N), the antisymmetric $k = 2$ representation has $B_{AS} = N - 8$ and $C_{AS} = 3$. For SU(5): $B_{10} = 5 - 8 = -3$. ✓

F.2 Core block index sums

The core block $\mathcal{C} = \mathbf{10} \oplus \mathbf{5} \oplus 2\bar{\mathbf{5}} \oplus 11 \cdot \mathbf{1}$ has:

Irreducible quartic.

$$B_{\mathcal{C}} = B_{10} + B_{\mathbf{5}} + 2 B_{\bar{\mathbf{5}}} + 11 \cdot 0 = (-3) + 1 + 2(1) + 0 = 0. \quad \checkmark \quad (210)$$

Cubic.

$$A_{3,\mathcal{C}} = A_3(\mathbf{10}) + A_3(\mathbf{5}) + 2 A_3(\bar{\mathbf{5}}) + 11 \cdot 0 = 1 + 1 + 2(-1) + 0 = 0. \quad \checkmark \quad (211)$$

Quadratic index.

$$A_{\mathcal{C}} = A_{10} + A_{\mathbf{5}} + 2 A_{\bar{\mathbf{5}}} + 11 \cdot 0 = 3 + 1 + 2(1) + 0 = 6 = L + 1. \quad \checkmark \quad (212)$$

Reducible quartic.

$$C_{\mathcal{C}} = C_{10} + C_{\mathbf{5}} + 2 C_{\bar{\mathbf{5}}} + 11 \cdot 0 = 3 + 0 + 0 + 0 = 3. \quad (213)$$

Dimension.

$$d_C = 10 + 5 + 2(5) + 11(1) = 10 + 5 + 10 + 11 = 36 = (L + 1)^2. \quad \checkmark \quad (214)$$

Dimension-to-index ratio.

$$\frac{d_C}{A_C} = \frac{36}{6} = 6 = L + 1. \quad \checkmark \quad (215)$$

This ratio is crucial: it ensures that the gravitational-gauge mixed coefficient $\text{coeff}(\text{tr } R^2 \cdot \text{Tr } F_5^2)$ has the correct value relative to the pure gauge coefficient $\text{coeff}((\text{Tr } F_5^2)^2)$ for the BI-6 factorization.

F.3 Anomaly polynomial of one core block

For a core block at chirality σ with charges (q_B, q_{Q_i}) , defining $x = q_B f_B + q_{Q_i} f_{Q_i}$, $a = \text{Tr } f_5^2$, and $g = \frac{1}{2}p_1$, the anomaly polynomial is:

$$\begin{aligned} I_8[\mathcal{C}^\sigma(q_B, q_{Q_i})] &= \sigma \left[\frac{d_C}{5760} (7p_1^2 - 4p_2) - \frac{d_C}{48} p_1 x^2 - \frac{A_C}{48} p_1 a \right. \\ &\quad \left. + \frac{d_C}{24} x^4 + \frac{A_C}{4} a x^2 + \frac{C_C}{24} a^2 + \underbrace{\frac{B_C}{24}}_{=0} \text{Tr } f_5^4 + \underbrace{\frac{A_{3,C}}{6}}_{=0} \text{Tr } f_5^3 x \right] \\ &= \sigma \left[\frac{36}{5760} (7p_1^2 - 4p_2) - \frac{36}{48} p_1 x^2 - \frac{6}{48} p_1 a + \frac{36}{24} x^4 + \frac{6}{4} a x^2 + \frac{3}{24} a^2 \right] \\ &= \sigma \left[\frac{1}{160} (7p_1^2 - 4p_2) - \frac{3}{4} p_1 x^2 - \frac{1}{8} p_1 a + \frac{3}{2} x^4 + \frac{3}{2} a x^2 + \frac{1}{8} a^2 \right]. \end{aligned} \quad (216)$$

The result is clean: the irreducible $\text{Tr } f_5^4$ and cubic $\text{Tr } f_5^3 x$ terms are *automatically absent* because $B_C = 0$ and $A_{3,C} = 0$. This is the core block's defining property: it pre-cancels the most dangerous anomaly monomials inside each block, before any inter-block cancellation is needed.

One block is clean; the question is how many of each kind to take. That is not model-building freedom but a Diophantine problem: four anomaly-matching equations in five orbit multiplicities. The next appendix solves it exactly, and the reader should watch for two things — how integrality forces $\Omega \in 25\mathbb{Z}$, and how an apparently negative multiplicity resolves into an honest chirality flip.

G The Diophantine system: derivation and solution

With the core-block anomaly data from Appendix F in hand, this appendix derives the Diophantine system that the orbit multiplicities must satisfy and exhibits the explicit solution stated in Theorem 7.8.

Physics intuition. The system that follows has five unknowns and four equations, so its solution set is a line, not a point — and the appendix says so explicitly, exhibiting the affine one-parameter family before selecting the self-similar $m = 0$ branch on which every multiplicity scales linearly with

$\Omega/25$. A student should absorb the division of labor: the *equations* come from anomaly matching and are non-negotiable; the *branch choice* is a minimality selection in the spirit of AX-4. Compare the string-side analogue: the Kumar–Morrison–Taylor enumeration of anomaly-free six-dimensional supergravities (Kumar, Morrison, and Taylor, 2010) also produces solution *families* from anomaly matching, with string universality conjectured to realize them; here the family is one-dimensional, explicit, and closed by a constitutional clause rather than by a landscape.

G.1 Orbit anomaly polynomials

Axial orbit A_n^σ . The axial orbit consists of two core blocks with charges $(n/3, 0)$ and $(0, n/3)$:

$$A_n^\sigma = \mathcal{C}^\sigma\left(\frac{n}{3}, 0\right) \oplus \mathcal{C}^\sigma\left(0, \frac{n}{3}\right). \quad (217)$$

For the first block, $x = (n/3)f_B$, so $x^2 = (n/3)^2 f_B^2$ and $x^4 = (n/3)^4 f_B^4$. For the second block, $x = (n/3)f_{Qi}$, giving the same with $B \leftrightarrow Qi$. The sum produces:

$$\begin{aligned} I_8[A_n^\sigma] = \sigma \bigg[& \frac{72}{5760} (7p_1^2 - 4p_2) - \frac{1}{4} p_1 a + \frac{1}{4} a^2 \\ & - \frac{n^2}{12} p_1 (f_B^2 + f_{Qi}^2) + \frac{n^2}{6} a (f_B^2 + f_{Qi}^2) + \frac{n^4}{54} (f_B^4 + f_{Qi}^4) \bigg]. \end{aligned} \quad (218)$$

Note that *no mixed $f_B^2 f_{Qi}^2$ or $f_B f_{Qi}$ terms appear*, because the two blocks have charges in orthogonal Abelian directions.

Diagonal orbit D_n^σ . The diagonal orbit consists of two core blocks with charges $(n/3, n/3)$ and $(n/3, -n/3)$:

$$D_n^\sigma = \mathcal{C}^\sigma\left(\frac{n}{3}, \frac{n}{3}\right) \oplus \mathcal{C}^\sigma\left(\frac{n}{3}, -\frac{n}{3}\right). \quad (219)$$

For the first block, $x = (n/3)(f_B + f_{Qi})$, so

$$x^2 = (n/3)^2 (f_B^2 + 2f_B f_{Qi} + f_{Qi}^2), \quad (220)$$

$$x^4 = (n/3)^4 (f_B^4 + 4f_B^3 f_{Qi} + 6f_B^2 f_{Qi}^2 + 4f_B f_{Qi}^3 + f_{Qi}^4). \quad (221)$$

For the second block, $x = (n/3)(f_B - f_{Qi})$, giving

$$x^2 = (n/3)^2 (f_B^2 - 2f_B f_{Qi} + f_{Qi}^2), \quad (222)$$

$$x^4 = (n/3)^4 (f_B^4 - 4f_B^3 f_{Qi} + 6f_B^2 f_{Qi}^2 - 4f_B f_{Qi}^3 + f_{Qi}^4). \quad (223)$$

When the two blocks are summed:

- The $f_B f_{Qi}$ terms in x^2 cancel: $+2 - 2 = 0$.
- The $f_B^3 f_{Qi}$ terms in x^4 cancel: $+4 - 4 = 0$.
- The $f_B f_{Qi}^3$ terms in x^4 cancel: $+4 - 4 = 0$.

- The diagonal terms f_B^2 and $f_{Q_i}^2$ in x^2 add: $1 + 1 = 2$.
- The $f_B^2 f_{Q_i}^2$ terms in x^4 add: $6 + 6 = 12$.
- The diagonal quartics f_B^4 and $f_{Q_i}^4$ add: $1 + 1 = 2$.

Therefore:

$$\begin{aligned}
 I_8[D_n^\sigma] = \sigma & \left[\frac{72}{5760} (7p_1^2 - 4p_2) - \frac{1}{4} p_1 a + \frac{1}{4} a^2 \right. \\
 & - \frac{n^2}{6} p_1 (f_B^2 + f_{Q_i}^2) + \frac{n^2}{3} a (f_B^2 + f_{Q_i}^2) \\
 & \left. + \frac{n^4}{27} (f_B^4 + f_{Q_i}^4) + \frac{2n^4}{9} f_B^2 f_{Q_i}^2 \right]. \tag{224}
 \end{aligned}$$

All off-diagonal terms have canceled by the $q_{Q_i} \rightarrow -q_{Q_i}$ symmetry.

G.2 Matching to the BI-6 target

Using the shorthand $b = f_B^2$, $c = f_{Q_i}^2$, $g = \frac{1}{2}p_1$, the target (excluding the pure gravitational g^2 term) is:

$$\frac{\Omega}{2} \left[-2ga + a^2 - \frac{4}{5}g(b+c) + \frac{4}{5}a(b+c) + \frac{4}{25}(b^2+c^2) + \frac{8}{25}bc \right]. \tag{225}$$

From the reservoir $\mathcal{M} = \bigoplus_n a_n A_n^+ \oplus \bigoplus_n d_n D_n^+$, the coefficients are:

$$\text{coeff}(ga) = -\frac{1}{2} \left(\sum_n a_n + \sum_n d_n \right) = -\Omega, \tag{226}$$

$$\text{coeff}(a^2) = \frac{1}{4} \left(\sum_n a_n + \sum_n d_n \right) = \frac{\Omega}{2}, \tag{227}$$

$$\text{coeff}(g(b+c)) = -\frac{1}{6} \sum_n n^2 a_n - \frac{1}{3} \sum_n n^2 d_n = -\frac{2\Omega}{5}, \tag{228}$$

$$\text{coeff}(a(b+c)) = \frac{1}{6} \sum_n n^2 a_n + \frac{1}{3} \sum_n n^2 d_n = \frac{2\Omega}{5}, \tag{229}$$

$$\text{coeff}(b^2+c^2) = \frac{1}{54} \sum_n n^4 a_n + \frac{1}{27} \sum_n n^4 d_n = \frac{2\Omega}{25}, \tag{230}$$

$$\text{coeff}(bc) = \frac{2}{9} \sum_n n^4 d_n = \frac{4\Omega}{25}. \tag{231}$$

Equations (226) and (227) both require

$$\sum_n a_n + \sum_n d_n = 2\Omega. \tag{232}$$

Equations (228) and (229) both require

$$\sum_n n^2 a_n + 2 \sum_n n^2 d_n = \frac{12\Omega}{5}. \tag{233}$$

Equation (231) gives

$$\sum_n n^4 d_n = \frac{18\Omega}{25}. \quad (234)$$

Equation (230) combined with (234):

$$\frac{1}{54} \sum_n n^4 a_n + \frac{1}{27} \cdot \frac{18\Omega}{25} = \frac{2\Omega}{25}, \quad (235)$$

so

$$\frac{1}{54} \sum_n n^4 a_n = \frac{2\Omega}{25} - \frac{18\Omega}{675} = \frac{54\Omega - 18\Omega}{675} = \frac{36\Omega}{675} = \frac{4\Omega}{75}, \quad (236)$$

hence

$$\sum_n n^4 a_n = \frac{4\Omega \times 54}{75} = \frac{216\Omega}{75} = \frac{72\Omega}{25}. \quad (237)$$

G.3 The solution

The linear system (232)–(234) has five unknowns $(a_1, a_2, d_1, d_2, d_3)$ and four equations, so one should expect an affine one-parameter family rather than a unique point. Solving the system exactly gives

$$a_1 = -\frac{64}{3}d_3 + \frac{152\Omega}{75}, \quad (238)$$

$$a_2 = \frac{4}{3}d_3 + \frac{4\Omega}{75}, \quad (239)$$

$$d_1 = \frac{77}{3}d_3 - \frac{2\Omega}{15}, \quad (240)$$

$$d_2 = -\frac{20}{3}d_3 + \frac{4\Omega}{75}. \quad (241)$$

To preserve integrality when $\Omega \in 25\mathbb{Z}$, write

$$\Omega = 25k, \quad d_3 = 2k + 3m, \quad k, m \in \mathbb{Z}.$$

Then the affine family becomes

$$a_1 = 8k - 64m, \quad (242)$$

$$a_2 = 4k + 4m, \quad (243)$$

$$d_1 = 48k + 77m, \quad (244)$$

$$d_2 = -12k - 20m, \quad (245)$$

$$d_3 = 2k + 3m. \quad (246)$$

Why the benchmark branch is chosen as $m = 0$. The benchmark used in the paper is the self-similar branch

$$m = 0, \quad d_3 = \frac{2\Omega}{25},$$

for which all coefficients scale linearly with $\Omega/25$:

$$\boxed{a_1 = \frac{8\Omega}{25}, \quad a_2 = \frac{4\Omega}{25}, \quad d_1 = \frac{48\Omega}{25}, \quad d_2 = -\frac{12\Omega}{25}, \quad d_3 = \frac{2\Omega}{25}.} \quad (247)$$

This is not the unique affine solution; it is the simplest scaling branch, and it is best to state this explicitly rather than leave it implicit.

Row-reduction in full. Starting from

$$a_1 + 16a_2 = \frac{72\Omega}{25}, \quad (248)$$

$$d_1 + 16d_2 + 81d_3 = \frac{18\Omega}{25}, \quad (249)$$

$$a_1 + 4a_2 + 2d_1 + 8d_2 + 18d_3 = \frac{60\Omega}{25}, \quad (250)$$

$$a_1 + a_2 + d_1 + d_2 + d_3 = \frac{50\Omega}{25}, \quad (251)$$

eliminate a_1 using (248):

$$a_1 = \frac{72\Omega}{25} - 16a_2.$$

Substituting into (250) and (251) gives

$$-12a_2 + 2d_1 + 8d_2 + 18d_3 = -\frac{12\Omega}{25}, \quad (252)$$

$$-15a_2 + d_1 + d_2 + d_3 = -\frac{22\Omega}{25}. \quad (253)$$

Double (253) and subtract from (252):

$$18a_2 + 6d_2 + 16d_3 = \frac{32\Omega}{25},$$

hence

$$3a_2 + d_2 + \frac{8}{3}d_3 = \frac{16\Omega}{75}. \quad (254)$$

Eliminate d_1 from (249):

$$d_1 = \frac{18\Omega}{25} - 16d_2 - 81d_3.$$

Substitute this into (253):

$$-15a_2 - 15d_2 - 80d_3 = -\frac{40\Omega}{25},$$

or

$$a_2 + d_2 + \frac{16}{3}d_3 = \frac{8\Omega}{75}. \quad (255)$$

Subtracting (255) from (254) yields

$$2a_2 - \frac{8}{3}d_3 = \frac{8\Omega}{75},$$

so

$$a_2 = \frac{4}{3}d_3 + \frac{4\Omega}{75}.$$

Then (255) gives

$$d_2 = -\frac{20}{3}d_3 + \frac{4\Omega}{75},$$

and back-substitution into (249) and (248) produces (240) and (238). This is the complete row-reduction.

Negative multiplicity is really opposite chirality. A potential confusion is that $d_2 < 0$ requires interpretation. The right interpretation is not negative particle number. Because chirality flips the sign of the six-dimensional anomaly polynomial,

$$I_8[D_n^-] = -I_8[D_n^+].$$

Therefore a negative coefficient multiplying D_n^+ is equivalent to a positive coefficient multiplying D_n^- :

$$d_2 D_2^+ = |d_2| D_2^- \quad \text{when } d_2 < 0.$$

On the benchmark branch (247), the physical reservoir can therefore be rewritten with nonnegative multiplicities as

$$\mathcal{M}_{\text{gauge}}^{\text{phys}}(\Omega) = \frac{8\Omega}{25}A_1^+ \oplus \frac{4\Omega}{25}A_2^+ \oplus \frac{48\Omega}{25}D_1^+ \oplus \frac{12\Omega}{25}D_2^- \oplus \frac{2\Omega}{25}D_3^+. \quad (256)$$

This resolves the physical concern cleanly: the reservoir contains no negative multiplicity, only a definite chirality pattern.

Minimum Ω_{orb} . The benchmark branch has coefficients proportional to $\Omega/25$, so integrality requires

$$\Omega_{\text{orb}} \in 25\mathbb{Z}.$$

At the first static benchmark $\Omega_{\text{orb}} = 25$,

$$(a_1, a_2, d_1, d_2, d_3) = (8, 4, 48, -12, 2),$$

equivalently

$$(8, 4, 48, 12, 2) \quad \text{with the fourth orbit of negative chirality.}$$

The gauge and mixed books now balance; gravity's remain. Pure gravitational anomalies cannot be traded against gauge choices — they see only the count and the chirality of what propagates — so what follows is the most rigid accounting in the paper: two curvature invariants, two species of field, one 2×2 linear system, and no freedom at all.

H Gravitational completion: detailed computation

The gauge-matter reservoir constructed in Appendix G satisfies all non-Abelian and mixed anomaly conditions. The purely gravitational sector remains: the p_1^2 and p_2 coefficients must also match. This appendix solves the 2×2 system for the number of gauge-singlet anti-self-dual tensors n_{SD} and singlet fermions n_F .

H.1 From the index theorem to the anomaly polynomials: a from-scratch derivation

The two per-field polynomials used below are quoted everywhere in the 6D literature but rarely derived in one place at graduate resolution; we derive them here. The textbook homes are [Alvarez-Gaumé and Witten \(1984\)](#) (the original), [Nakahara \(2003\)](#) (Ch. 12, index theorems), and [Bilal \(2008\)](#) (§5); the string-side deployment is [Green, Schwarz, and Witten \(1987\)](#) (Ch. 13).

Step 1: the two genera. The gravitational anomaly of a chiral field is governed by an index density: the $\hat{A}$ -genus for spin-1/2 fermions and the Hirzebruch L -genus for (anti-)self-dual tensors. Both are characteristic classes built from the curvature: a genus assigns to a manifold a polynomial in its Pontryagin classes whose integral computes the index of a differential operator on it. The $\hat{A}$ -genus does this for the Dirac operator and the L -genus for the signature operator, and that is the whole of what is used below. Their expansions in Pontryagin classes are

$$\hat{A} = 1 - \frac{p_1}{24} + \frac{7p_1^2 - 4p_2}{5760} + \dots, \quad (257)$$

$$L = 1 + \frac{p_1}{3} + \frac{7p_2 - p_1^2}{45} + \dots. \quad (258)$$

Both follow from the defining products $\hat{A} = \prod_i \frac{x_i/2}{\sinh(x_i/2)}$ and $L = \prod_i \frac{x_i}{\tanh x_i}$ by expanding to fourth order in the Chern roots x_i and re-expressing the symmetric polynomials through $p_1 = \sum x_i^2$ and $p_2 = \sum_{i < j} x_i^2 x_j^2$; the $5760 = 2^7 \cdot 3^2 \cdot 5$ is the honest denominator of the fourth-order sinh expansion, and the student should carry it through once by hand.

Step 2: the fermion eight-form. For a positive-chirality Weyl fermion in six dimensions, the anomaly polynomial is the degree-eight term of $\hat{A}$ (times the gauge character, set to one for a singlet):

$$I_8^F = [\hat{A}]_8 = \frac{7p_1^2 - 4p_2}{5760}. \quad (259)$$

Step 3: the tensor eight-form. For one chiral two-form, the contribution is $-\frac{1}{8}$ of the degree-eight term of the L -genus, the normalization fixed by the signature theorem for the (anti-)self-dual field strength ([Alvarez-Gaumé and Witten, 1984](#)):

$$I_8^T = -\frac{1}{8}[L]_8 = -\frac{1}{8} \cdot \frac{7p_2 - p_1^2}{45} = \frac{p_1^2 - 7p_2}{360} = \frac{16p_1^2 - 112p_2}{5760}. \quad (260)$$

Physics intuition. Why does one tensor “count as twenty-eight fermions”? Compare the p_2 coefficients of (259) and (260): $-112/5760$ against $-4/5760$, a ratio of exactly 28. Add the one Weyl fermion that accompanies the tensor in a supergravity tensor multiplet and one obtains the famous 29 of the $n_H - n_V + 29 n_T = 273$ constraint (84) — which the student can now regard as an exercise rather than an incantation. The deeper point is the *ratio mismatch*: the fermion produces p_2 and p_1^2 in the fixed ratio $-4 : 7$, the tensor in the ratio $-7 : 1$; because the target polynomial $(\Omega/2)X_4^2$ contains no irreducible p_2 at all, no quantity of fermions alone can land on it, and the two-species 2×2 system solved below is forced. The same mismatch, one dimension pair up, is what forces the gravitino’s 496 in the ten-dimensional mechanism (Appendix C): an irreducible curvature monomial always dictates a spectrum count.

H.2 Anomaly polynomials for the gravitational sector

Weyl fermion. For one positive-chirality gauge-singlet Weyl fermion, the gravitational anomaly polynomial is (from the $\hat{A}$ -genus):

$$I_8^F = \frac{7p_1^2 - 4p_2}{5760}. \quad (261)$$

Chiral tensor. For one real chiral two-form tensor, the gravitational anomaly comes from the Hirzebruch L -genus (Alvarez-Gaumé and Witten, 1984). For the tensor chirality aligned with the positive-chirality fermion convention above, the contribution is

$$I_8^T = \frac{16p_1^2 - 112p_2}{5760} = \frac{p_1^2 - 7p_2}{360}, \quad (262)$$

and the opposite (anti-self-dual) chirality contributes $-I_8^T$. The ratio of the tensor’s p_2 coefficient to the fermion’s is $-112/-4 = 28$; together with the one Weyl fermion of a tensor multiplet, this is the origin of the familiar factor 29 in the supergravity constraint (84). The tensor’s p_2/p_1^2 ratio -7 differs from the fermion’s $-4/7$; this difference is what breaks the universal ratio and allows the gravitational system to be solved.

H.3 The gravitational deficit

The gauge-matter reservoir at normalization Ω contributes 144Ω effective fermion dimensions. Its gravitational polynomial is:

$$I_8^{\text{gauge,grav}} = \frac{144\Omega}{5760}(7p_1^2 - 4p_2) = \frac{\Omega}{40}(7p_1^2 - 4p_2). \quad (263)$$

The target is

$$I_8^{\text{target,grav}} = \frac{\Omega}{2} \cdot \frac{p_1^2}{4} = \frac{\Omega p_1^2}{8}, \quad (264)$$

since the gravitational slot of X_4 is $p_1/2$ in the characteristic-class normalization fixed by the mixed-sector matching of Appendix G and by the integral-lift lemma of Appendix I. The deficit is

$$\Delta I_8 = \frac{\Omega p_1^2}{8} - \frac{\Omega}{40}(7p_1^2 - 4p_2) = \Omega \left(\frac{p_1^2}{8} - \frac{7p_1^2}{40} + \frac{p_2}{10} \right) = \Omega \left(-\frac{p_1^2}{20} + \frac{p_2}{10} \right). \quad (265)$$

H.4 The 2×2 system

We need

$$n_F I_8^{F,-} + n_{\text{SD}} I_8^{\text{ASD}} = \Delta I_8, \quad (266)$$

where both completion species are gauge singlets: n_{SD} anti-self-dual tensors, each contributing $I_8^{\text{ASD}} = -\frac{1}{5760}(16p_1^2 - 112p_2)$ (the chirality opposite to (262)), and n_F Weyl fermions of chirality opposite to the gauge reservoir, each contributing $I_8^{F,-} = -\frac{1}{5760}(7p_1^2 - 4p_2)$. Matching the p_1^2 and p_2 coefficients of the deficit (265) in units of $1/5760$:

$$-7 n_F - 16 n_{\text{SD}} = -288 \Omega, \quad (267)$$

$$+4 n_F + 112 n_{\text{SD}} = 576 \Omega. \quad (268)$$

Equation (268) simplifies to $n_F + 28 n_{\text{SD}} = 144 \Omega$, hence $n_F = 144 \Omega - 28 n_{\text{SD}}$. Substituting into (267):

$$-7(144 \Omega - 28 n_{\text{SD}}) - 16 n_{\text{SD}} = -288 \Omega \implies 180 n_{\text{SD}} = 720 \Omega, \quad (269)$$

so that

$$\boxed{n_{\text{SD}} = 4 \Omega, \quad n_F = 144 \Omega - 112 \Omega = 32 \Omega.} \quad (270)$$

Both counts are positive integers for every $\Omega \in 25\mathbb{Z}$, linear in Ω , and they satisfy the exact structural ratio

$$\frac{n_F}{n_{\text{SD}}} = 8 : \quad (271)$$

exactly eight opposite-chirality singlet Weyl fermions accompany each anti-self-dual tensor, independently of Ω . The chirality flip of the completion sector relative to the gauge reservoir is the same bookkeeping as the D_2^- orbit of the Diophantine solution (Appendix G): the reservoir over-produces the p_1^2 coefficient relative to the target ($7\Omega/40 > \Omega/8$), so the completion must enter with the opposite sign of both gravitational coefficients.

Accounting for the Green–Schwarz tensor itself. The count $n_{\text{SD}} = 4 \Omega$ is the *total* chiral-tensor content required by the gravitational ledger, and it includes the Green–Schwarz mediator B_2 itself, which is anti-self-dual at the quantum level (Proposition 5.10) and carries the same gravitational anomaly as each spectator. The X_4 -coupled channel remains unique and primitive (the tensor clause of Definition 6.1); the remaining $4\Omega - 1$ chiral tensors are gauge-singlet spectators carrying gravitational anomaly only. At the minimal branch $\Omega = 25$: one mediator plus 99 spectators.

Verification. In units of $1/5760$, at general Ω : the p_1^2 coefficient is $-7(32 \Omega) - 16(4 \Omega) = -288 \Omega = 5760 \times (-\Omega/20) \checkmark$; the p_2 coefficient is $+4(32 \Omega) + 112(4 \Omega) = 576 \Omega = 5760 \times (\Omega/10) \checkmark$.

For the minimal benchmark $\Omega = 25$:

$$n_{\text{SD}} = 100, \quad n_F = 800, \quad (272)$$

and for $\Omega = 50$: $n_{\text{SD}} = 200$, $n_F = 1600$. At the minimal branch $\Omega = L^2 = 25$ the tensor count is the perfect square $n_{\text{SD}} = 4L^2 = (2L)^2 = 100$; whether this reflects a deeper OGN structure is an open question. Because $n_{\text{SD}} = 4\Omega$, the tensor count is divisible by 8 exactly when Ω is even: the $\Omega = 50$ branch is manifestly signature-friendly (Appendix I), while on the $\Omega = 25$ branch the tensor-sector phase vanishes identically on the intended-class fillings, where $\sigma(X_6 \times D^2) = 0$ (Lemma I.10).

Everything so far is local: the ledger balances point by point on the bulk. But an anomaly can hide where no local computation looks — in the holonomy the theory picks up when carried around a loop of backgrounds, an eigenvalue slipping through zero far from any Feynman diagram. The final constructive appendix chases that last hiding place with the eta invariant, and closes it on the intended class.

I Global anomaly proof and automatic admissibility

The preceding appendices have established the local anomaly cancellation: the descent proof (B), the rank-one factorization and coefficient matching (D), the OGN-5 derivation (E), the core block and Diophantine spectrum (F–G), and the gravitational completion (H). Together they discharge proof obligations (PO1)–(PO6) of Section 5.4.

The present appendix addresses the remaining obligations (PO7)–(PO8): global gauge anomalies and the well-definedness of the exponentiated Green–Schwarz term on the charge lattice. The proof upgrades the local BI-6 cancellation to a full global theorem on the admissible GG-6 background class.

I.1 A primer on the eta invariant and the Dai–Freed formalism (for the graduate reader)

The global analysis below uses machinery that most graduate curricula postpone; this primer supplies the working minimum. The modern references are Witten (2016) (the physicist’s treatment), García-Etxebarria and Montero (2019) (the same machinery deployed on particle-physics gauge groups), Dai and Freed (1994) (the original determinant-line construction), and Nakahara (2003) (Ch. 12) for the index-theory background.

The eta invariant. For a self-adjoint Dirac operator D on a closed odd-dimensional manifold, define

$$\eta(s) = \sum_{\lambda \neq 0} \text{sign}(\lambda) |\lambda|^{-s}, \quad \eta \equiv \eta(0)_{\text{reg}}, \quad (273)$$

the regularized spectral asymmetry: it counts how lopsided the spectrum is between positive and negative eigenvalues. The Atiyah–Patodi–Singer theorem ties it to an index on a bounding manifold:

for $\partial W = U$,

$$\text{index } D_W = \int_W \hat{A} \wedge \text{ch}(F) - \frac{\eta(U) + h(U)}{2}, \quad (274)$$

with h the number of zero modes. Rearranged, this is exactly the statement used by Lemma I.10: the combination $\int_W I - \xi(U)$ (with $\xi = (\eta + h)/2$) is an *integer*, so its exponential is trivial.

A one-line worked example. Take the free fermion on a circle with Wilson-line twist θ : eigenvalues $\lambda_n = n + \theta/2\pi$, $n \in \mathbb{Z}$. A short computation gives $\eta = 1 - \theta/\pi$ (mod even integers) for $0 < \theta < 2\pi$: as the twist winds once, one eigenvalue flows through zero and η jumps by 2. This *spectral flow* in miniature is the entire content of the global anomaly: transporting the theory around a loop in background space can slide eigenvalues through zero, and each crossing flips the sign of the fermion determinant. The mapping torus construction below is this example writ large — the twist θ becomes a large gauge or diffeomorphism transformation, the circle becomes the loop in the space of backgrounds, and $\exp(2\pi i \xi)$ is the accumulated phase.

The Dai–Freed prescription. On manifolds with boundary the chiral partition function is not a number but an element of a determinant *line*; Dai and Freed (1994) showed that $\exp(2\pi i \xi(U))$ of a bounding U supplies the canonical trivialization, making the combined bulk-boundary object well defined whenever the APS combination (274) is used. The practical recipe for the physicist, and the one Appendix sections below implement: to test a large transformation, build its mapping torus U_7 , fill it with any W_8 carrying the extended data, and evaluate $\exp[2\pi i (\int_{W_8} I_8 - \xi(U_7))]$; consistency requires the answer to be filling-independent (the Wu-shift lemma) and equal to one (the cancellation lemmas).

Physics intuition. The local anomaly is a statement about the *curvature* of the anomaly line over the space of backgrounds; the global anomaly is a statement about its *holonomy*. Green–Schwarz cancellation flattens the line — that is the local theorem. A flat line can still have holonomy around non-contractible loops — that is the residual global question, and it is answered on the intended class by Lemma I.10 (every allowed loop is contractible, so the flat line has no holonomy to show) and routed to Open Problem O10 off it.

I.2 What exactly has to be proved?

The phrase *complete anomaly freedom* means more than the statement that the descent polynomial vanishes locally. For a six-dimensional chiral theory one must distinguish three layers.

Layer 1. Infinitesimal / local anomaly. Under small gauge or local Lorentz transformations the one-loop effective action varies by the six-form descent term $I_6^{(1)}$. BI-6 cancels this part exactly if the anomaly polynomial factorizes as $I_8 = \frac{1}{2} X_4^2$.

Layer 2. Large gauge / diffeomorphism anomaly. Around a non-contractible loop in the space of backgrounds, the partition function can return with a nontrivial phase even when the local descent variation vanishes.

Layer 3. Self-dual / anti-self-dual tensor global phase. In six dimensions a chiral two-form is not described globally by the naive local action $\int B_2 \wedge X_4$; it needs a quadratic refinement / Wu-shifted differential-cocycle definition.

In other words, the target theorem is:

for every admissible large transformation of every admissible GG-6 background, the total exponentiated path integral returns with phase one.

That is stronger than the local theorem, and the distinction is essential.

Dimensional ladder. The correct dimensional hierarchy for the global anomaly is:

$$\text{6D theory on } M_6 \implies \text{7D mapping torus } U_7 \implies \text{8D filling } W_8.$$

Large gauge or diffeomorphism transformations of the 6D theory are encoded by the closed mapping torus U_7 . To evaluate the anomaly phase one chooses an 8-manifold W_8 with $\partial W_8 = U_7$ and extends the spin/gauge/tensor data to W_8 . That is the correct modern APS/Dai–Freed setting.

In short: the global anomaly is a seven-dimensional holonomy problem and an eight-dimensional filling problem, not a direct six-dimensional descent problem.

The local descent proof is given in full in Appendix B above.

I.3 Admissible background class and hypotheses H1–H4

To close the global proof cleanly, we must state the precise hypotheses. We label them $H1$ – $H4$ and sharpen each one.

Definition I.1 (Admissible GG-6 background class). An admissible background consists of:

- (a) a compact Euclideanized six-manifold M_6 obtained either by doubling the interval geometry $M_4 \times I_r \times S^1_\theta$ or by imposing APS-compatible boundary conditions at the ends of I_r ;
- (b) a spin structure on M_6 ;
- (c) principal bundles for $SU(5)$, $U(1)_B$, and $U(1)_{Qi}$;
- (d) a tensor-sector differential cocycle $\check{B}$ whose modified field strength has curvature representative X_4 .

This abstract definition gives the minimal data needed to formulate the conditional global theorem below. The concrete physical realization is the nine-clause class **Admissible** defined in Definition 6.1 of the main text; the automatic admissibility lemmas in Section I.8 below prove that every background in **Admissible** satisfies hypotheses (H1)–(H4) automatically.

- (H1) **Bordism / extension hypothesis.** For every admissible large gauge or diffeomorphism transformation of an admissible background, the associated mapping torus U_7 represents the trivial class in the relevant spin-gauge bordism group, so there exists a spin eight-manifold W_8 with $\partial W_8 = U_7$ to which the gauge and tensor data extend.

- (H2) **No independent Witten-type gauge obstruction.** The gauge group is exactly $SU(5) \times U(1)_B \times U(1)_{Qi}$, and

$$\pi_6(SU(5)) = 0, \quad \pi_6(U(1)_B) = 0, \quad \pi_6(U(1)_{Qi}) = 0.$$

Hence there is no separate six-dimensional homotopy anomaly of the Witten type.

- (H3) **Tensor-lattice / quadratic-refinement hypothesis.** The lattice-valued lift of the BI-6 four-form obeys the correct shifted integrality law, equivalently the exponentiated Wu–Chern–Simons functional is single-valued. On every closed eight-manifold Z_8 ,

$$\frac{1}{2} \int_{Z_8} (X_\Lambda \cup X_\Lambda - X_\Lambda \cup \lambda) \in \mathbb{Z}, \quad (275)$$

where X_Λ is the lattice-valued lift of the BI-6 cocycle and λ is the appropriate Wu / characteristic lift.

- (H4) **Differential-cocycle hypothesis.** The factorizing four-form X_4 lifts to a globally defined degree-4 differential cocycle $\check{X}$ on all admissible mapping tori and their fillings. This is the global version of the local identity $dH_3 = X_4$.

Remark I.2. (H1) and (H4) are the global geometric hypotheses. (H2) removes an independent gauge-homotopy obstruction. (H3) is the crucial quantum statement: it is what makes the chiral two-form / Green–Schwarz functional globally well-defined instead of merely locally meaningful.

I.4 The anomaly line and the mapping-torus formulation

Let $\mathcal{B}$ be the space of admissible GG-6 background fields and $\mathcal{G}$ the group of gauge / local Lorentz / diffeomorphism transformations. The chiral matter partition function is not a function on $\mathcal{B}/\mathcal{G}$; it is a section of an anomaly line bundle

$$\mathcal{L}_{\text{an}} \longrightarrow \mathcal{B}/\mathcal{G}.$$

Its curvature is the local anomaly polynomial, while its holonomy around a loop is the global anomaly phase.

A large transformation defines a loop $\gamma \subset \mathcal{B}/\mathcal{G}$. The corresponding global anomaly is

$$\text{hol}_{\mathcal{L}_{\text{an}}}(\gamma) = \mathcal{A}_\chi(U_7), \quad (276)$$

where U_7 is the mapping torus of the 6D background under that transformation.

The APS / Dai–Freed form of the anomaly phase can be written schematically as

$$\mathcal{A}_\chi(U_7) = \exp \left[2\pi i \left(\int_{W_8} I_8 - \xi_\chi(U_7) \right) \right]. \quad (277)$$

Here W_8 is any filling of U_7 , and $\xi_\chi(U_7)$ denotes the reduced eta-invariant / torsion correction carried by the seven-dimensional boundary operator. The exact normalization of ξ_χ is not the

important issue here. The important issue is that the global anomaly contains *two* kinds of terms:

1. an integral $\int_{W_8} I_8$, which is the extension of the local descent data;
2. a torsion / eta piece ξ_X , which is invisible to the naive local formula $\int B_2 \wedge X_4$.

This is exactly why the global Green–Schwarz term must be written as a differential-cohomology / Wu–Chern–Simons object.

I.5 The globally defined BI-6 counterterm

Why the local expression is not enough. The local expression

$$\frac{1}{2} \int_{M_6} B_2 \wedge X_4$$

is perfectly adequate on a topologically trivial patch. But under large gauge transformations or on a manifold with nontrivial cohomology it is not, by itself, a globally defined number. The chiral two-form is a higher gauge field, not an ordinary globally defined 2-form everywhere.

The correct global object is a *Wu–Chern–Simons* / differential-cohomology functional on the mapping torus.

Definition of the global phase. Let $\check{X}$ be the degree-4 differential cocycle lifting X_4 . On a closed mapping torus U_7 bounding W_8 , define

$$S_{\text{WCS}}(U_7; \check{X}) := \frac{1}{2} \int_{W_8} (\check{X} \cup \check{X} - \check{X} \cup \check{\lambda}) \in \mathbb{R}/\mathbb{Z}. \quad (278)$$

The exponentiated BI-6 counterphase is then

$$\mathcal{A}_{\text{GS}}(U_7) := \exp[-2\pi i S_{\text{WCS}}(U_7; \check{X})]. \quad (279)$$

Equation (278) is the global replacement for the local BI-6 term. The cup-product notation is deliberate: global anomaly cancellation is a statement in differential cohomology, not only in differential forms.

Remark I.3. On topologically simple backgrounds, or on an admissible class where the Wu shift trivializes, (278) reduces to the familiar form

$$\mathcal{A}_{\text{GS}}(U_7) \sim \exp\left[-2\pi i \frac{1}{2} \int_{W_8} X_4^2\right].$$

This is the fully global formula. The full $\check{\lambda}$ -shift is what makes the expression globally meaningful.

I.6 Detailed proof of global anomaly cancellation

We now prove the theorem rigorously on the admissible class.

Theorem I.4 (Complete BI-6 anomaly freedom on the admissible class). *Assume the admissible GG-6 background class satisfies (H1)–(H4). Then for every admissible large gauge transformation, local Lorentz transformation, or diffeomorphism, the combined holonomy of the chiral determinant and the exponentiated BI-6 counterterm is trivial:*

$$\mathcal{A}_{\text{tot}}(U_7) := \mathcal{A}_\chi(U_7) \mathcal{A}_{\text{GS}}(U_7) = 1. \quad (280)$$

Hence BI-6 is completely anomaly-free on the admissible GG-6 background class.

The proof will be split into four lemmas.

Lemma 1: the BI-6 global counterterm is well-defined

Lemma I.5. *Under (H1), (H3), and (H4), the phase $\mathcal{A}_{\text{GS}}(U_7)$ defined in (279) is independent of the chosen filling W_8 .*

Proof. Suppose W_8 and W'_8 are two fillings of the same mapping torus U_7 , both carrying the extended cocycle $\check{X}$. Glue them along the common boundary with reversed orientation to obtain a closed eight-manifold

$$Z_8 = W_8 \cup_{U_7} (-W'_8).$$

The ratio of the two exponentiated actions is

$$\frac{\exp[-2\pi i S_{\text{WCS}}(U_7; \check{X})]_{W_8}}{\exp[-2\pi i S_{\text{WCS}}(U_7; \check{X})]_{W'_8}} = \exp\left[-2\pi i \frac{1}{2} \int_{Z_8} (X_\Lambda \cup X_\Lambda - X_\Lambda \cup \lambda)\right]. \quad (281)$$

By hypothesis (H3), the exponent is an integer. Therefore the ratio is one. So the phase is independent of the filling and is globally well-defined. ■

Lemma 2: the non-torsion integral part cancels exactly

Lemma I.6. *Under $I_8 = \frac{1}{2}X_4^2$, the extension-dependent integral part of the chiral anomaly phase is canceled exactly by the extension-dependent integral part of the BI-6 counterterm.*

Proof. The chiral determinant contributes the factor

$$\exp\left[2\pi i \int_{W_8} I_8\right] = \exp\left[2\pi i \int_{W_8} \frac{1}{2}X_4^2\right].$$

The global BI-6 counterterm contributes

$$\exp\left[-2\pi i \int_{W_8} \frac{1}{2}X_4^2\right] \times \exp[\text{Wu/torsion correction}].$$

Multiplying the first factors gives one. Thus the entire non-torsion integral contribution cancels identically. This is the global lift of the local factorization theorem. ■

Lemma 3: there is no independent Witten-type gauge obstruction

Lemma I.7. *Under (H2), there is no independent six-dimensional Witten-type global gauge anomaly for the GG-6 gauge group.*

Proof. For a six-dimensional chiral gauge theory, an independent Witten-type global gauge anomaly is classified by a homotopy class in $\pi_6(G)$. In GG-6,

$$G = \mathrm{SU}(5) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_{Qi}.$$

Now Bott periodicity gives $\pi_6(\mathrm{SU}(n)) = 0$ for $n \geq 4$, hence

$$\pi_6(\mathrm{SU}(5)) = 0.$$

For $\mathrm{U}(1)$, one has $\pi_k(\mathrm{U}(1)) = 0$ for all $k > 1$, so

$$\pi_6(\mathrm{U}(1)_B) = 0, \quad \pi_6(\mathrm{U}(1)_{Qi}) = 0.$$

Because the homotopy group of a direct product is the direct sum of the homotopy groups of the factors, $\pi_6(G) = 0$. Hence there is no independent Witten-type gauge phase beyond the anomaly-field-theory phase already encoded by the mapping torus formalism. ■

Lemma 4: the eta-invariant / tensor-sector phase is exactly the quadratic-refinement phase

Lemma I.8. *Under (H3) and (H4), the residual eta-invariant / torsion phase of the chiral determinant is exactly the phase canceled by the Wu–Chern–Simons / self-dual-tensor quadratic refinement.*

Proof. The local form $\frac{1}{2} \int B_2 \wedge X_4$ knows only about the de Rham class of X_4 . The missing data are the torsion and half-integral shifts that arise for a chiral two-form. These are encoded mathematically by the quadratic refinement of the cup product. In modern language, the anomaly field theory of the self-dual tensor is a Wu–Chern–Simons theory; its partition function on U_7 carries precisely the quadratic-refinement phase that trivializes the Dai–Freed eta phase of the chiral matter sector.

Concretely, hypothesis (H3) says that the lattice-valued cocycle X_Λ is chosen so that

$$\frac{1}{2} \int_{Z_8} (X_\Lambda \cup X_\Lambda - X_\Lambda \cup \lambda) \in \mathbb{Z}$$

on all closed eight-manifolds. This is exactly the integrality condition that makes the quadratic refinement single-valued. Therefore the residual phase of the self-dual / anti-self-dual tensor sector is not an extra uncanceled anomaly; it is the missing piece required to cancel the eta-invariant contribution of the chiral determinant. Hence the residual torsion phase cancels. ■

Remark I.9 (Methodological scope of Lemma 4). The argument uses the general Hopkins–Singer / Monnier–Moore framework ([Hopkins and Singer, 2005](#), [Monnier and Moore, 2019](#)) at the structural

level: the quadratic-refinement machinery guarantees cancellation whenever $(H3)$ and $(H4)$ hold. What is not done in this paper is an explicit computation of the anomaly field theory’s partition function on every relevant mapping torus of the non-supersymmetric BI-6 spectrum—the kind of case-by-case verification that Monnier and Moore carry out for $\mathcal{N} = (1, 0)$ SUGRA. For the intended GG-6 class, this additional verification is not strictly required, because the admissible mapping tori are topologically simple enough (product structure, null-homotopic clutching, simply connected base) that the structural argument is sufficient. A fully explicit non-SUSY anomaly-field-theory computation, parallel to the Monnier–Moore treatment but without a multiplet structure, would constitute a valuable independent confirmation and a productive direction for collaboration with the six-dimensional anomaly community.

Lemma I.10 (Computed phase on the intended class). *On Admissible, the combined global phase of every allowed large transformation equals 1 exactly.*

Proof. Two independent routes. *Route A (flat holonomy).* The combined anomaly line $\mathcal{L}_{\text{an}} \otimes \mathcal{L}_{\text{GS}}$ over the space of backgrounds is globally defined (Lemma 1) and *flat*: its curvature is the local anomaly two-form, which vanishes identically by the exact Green–Schwarz cancellation of Theorem 5.5. Every allowed transformation is isotopic to the identity (clause (C6)), so the loop it defines in the space of backgrounds is contractible, the isotopy providing the contracting disk over which the background data extend (product/collar clauses). The holonomy of a flat line around a contractible loop is 1. *Route B (APS index on the product filling).* The doubled mapping torus factorizes as $U_7 \cong X_6 \times S^1$ with bounding structure and bounds $W_8 = X_6 \times D^2$ with all data extending (Lemma for H1). For each chiral fermion species, the Atiyah–Patodi–Singer index theorem gives $\int_{W_8} I_8^{(s)} - \xi_s(U_7) = \text{index } D_{W_8}^{(s)} \in \mathbb{Z}$, so its Dai–Freed factor is $\exp(2\pi i \cdot \mathbb{Z}) = 1$. For the Green–Schwarz sector, the Wu–Chern–Simons functional is filling-independent (Lemma 1) and its cocycle on W_8 is a pullback from X_6 , so no torsion pairing survives. For the chiral-tensor sector, $\sigma(X_6 \times D^2) = 0$: every class in $H^4(W_8; \mathbb{R})$ is a pullback $\pi^* \alpha$ from X_6 , and $\pi^* \alpha \cup \pi^* \beta = \pi^*(\alpha \cup \beta)$ with $\alpha \cup \beta \in H^8(X_6) = 0$; the relative intersection form therefore vanishes identically, and the signature-type phase $\exp(2\pi i n_{\text{SD}} \sigma(W_8)/8)$ equals 1 at every $\Omega \in 25\mathbb{Z}$ — including the AX-4-minimal branch $\Omega = 25$, where the mod-8 shortcut is unavailable. The two routes agree, and no residual phase remains. ■

Remark I.11. Lemma I.10 upgrades Lemmas 2–4 from structural to computed on the intended class: the eta content of the Dai–Freed formula is evaluated (it is an integer index for every allowed loop), and the tensor phase is evaluated (the filling signature vanishes). The complement — the winding, mixed-clutching, and non-isotopic classes excluded by clause (C6) — is now covered by Theorem 6.9 on signature-normalized fillings; what survives of Open Problem O10 is recorded in its own entry in Section 17.

Completion of the proof of Theorem I.4

Proof of Theorem I.4. Take any admissible large gauge/diffeomorphism transformation and form its mapping torus U_7 .

1. By Lemma 1, the exponentiated BI-6 counterterm is globally well-defined on U_7 .
2. By Lemma 2, the extension-dependent integral part of the chiral determinant cancels exactly against the extension-dependent integral part of the BI-6 counterterm because $I_8 = \frac{1}{2}X_4^2$.
3. By Lemma 3, there is no independent Witten-type six-dimensional gauge obstruction because $\pi_6(G) = 0$.
4. By Lemma 4, the residual eta-invariant / torsion phase is precisely the quadratic-refinement phase canceled by the Wu–Chern–Simons definition of the tensor sector.

Therefore no residual phase remains. Hence

$$\mathcal{A}_{\text{tot}}(U_7) = 1$$

for every admissible mapping torus. The anomaly line has trivial holonomy on the admissible class, so the combined exponentiated path integral is globally well-defined and gauge invariant. This is complete anomaly freedom on the admissible GG-6 background class. ■

The four lemmas exhaust all known sources of global anomaly in six dimensions: the extension-dependent integral (Lemma 2), independent Witten-type homotopy (Lemma 3), and the residual eta/torsion phase (Lemma 4), with Lemma 1 ensuring that the counterterm is well-defined in the first place.

I.7 Specialization, scope, and reference data

The GG-6 gauge group and the vanishing of π_6 The GG-6 gauge group is constitutionally fixed to

$$\text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Qi}.$$

This is unusually favorable for the global-anomaly proof, because the dangerous six-dimensional homotopy groups vanish exactly as required in Lemma 3.

Why the BI-6 coefficients are compatible with integrality. In integral characteristic-class normalization, define

$$\frac{X_4}{8\pi^2} = \frac{1}{2}p_1(TM_6) - c_2(E_5) - \frac{2}{5}c_1(L_B)^2 - \frac{2}{5}c_1(L_{Qi})^2. \quad (282)$$

Because M_6 is spin, $\lambda := p_1/2 \in H^4(M_6, \mathbb{Z})$. Therefore

$$5\frac{X_4}{8\pi^2} = 5\lambda - 5c_2(E_5) - 2c_1(L_B)^2 - 2c_1(L_{Qi})^2 \in H^4(M_6, \mathbb{Z}). \quad (283)$$

So the BI-6 form is not arbitrarily fractional. Its only denominator is exactly the denominator that is killed by multiplying by five.

This is why the present paper naturally arrives at the arithmetic condition

$$\Omega \in 25\mathbb{Z}.$$

The factor of 25 is the square of the denominator that must be absorbed when one exponentiates the quadratic GS/WCS action.

The particularly clean branch $\Omega = 50$ (The AX-4-canonical benchmark of the main text is $\Omega = 25$; the $\Omega = 50$ branch is kept here because its signature arithmetic is the cleanest to display.) This appendix works the branch with

$$\Omega = 50, \quad n_{SD} = 200, \quad n_F = 1600.$$

For this benchmark the model-level anomaly polynomial is

$$I_8^{\text{tot}} = \frac{50}{2} X_4^2 = 25 X_4^2.$$

Since $5X_4$ is integral by (283), the non-torsion part is especially transparent:

$$25X_4^2 = (5X_4)^2.$$

So the form-level integrality is manifest.

The same benchmark also makes the pure signature part of the self-dual completion especially clean. A single self-dual tensor carries a signature-type phase of the schematic form

$$\exp\left(2\pi i \frac{\sigma(W_8)}{8}\right).$$

For 200 chiral tensors this becomes

$$\exp\left(2\pi i 200 \frac{\sigma(W_8)}{8}\right) = \exp(2\pi i 25 \sigma(W_8)) = 1.$$

So the benchmark chosen in the present paper is not only locally consistent; it is also globally friendly.

The values $\Omega = 50$ and $n_{SD} = 200$ are a concrete explicit branch on which the global anomaly phase closes cleanly; the AX-4-canonical benchmark of the main text is $\Omega = 25$, and the minimal solution is Open Problem O8.

What has now really been proved. This appendix proves a complete cancellation theorem *on the admissible background class*. That is already a real theorem and not merely a heuristic. But it is equally important to state what remains open.

When this appendix was first written, three items remained outside its scope; the main text has since closed two of them, and the record should say so. The bordism triviality of every closed spin seven-manifold with G -bundle — not just the intended-class mapping tori — is now Theorem 6.9 via

Lemma 6.6; and uniqueness of the tensor lattice’s Wu–Chern–Simons refinement is now Theorem 6.13. What remains open of the original three is the middle item: a fully self-contained exposition of the differential-cocycle / Wu-shift formalism within the paper, beyond the structural level at which it is applied here.

Self-contained theorem formulation. Stated in a form independent of the surrounding construction, the theorem reads as follows.

Theorem I.12 (Self-contained BI-6 global-anomaly theorem). *For the GG-6 bulk with gauge group $SU(5) \times U(1)_B \times U(1)_{Q_i}$ and BI-6 four-form X_4 satisfying $I_8 = \frac{1}{2}X_4^2$, the perturbative local anomaly is canceled exactly by the Green–Schwarz mechanism. Moreover, on the admissible class of spin mapping-torus backgrounds for which the gauge/tensor data extend to eight-manifold fillings and the BI-6 cocycle satisfies the required shifted integrality / quadratic-refinement condition, the combined phase of the chiral determinant and the exponentiated BI-6 counterterm is globally well-defined and gauge invariant. Hence BI-6 is completely anomaly-free on that admissible class.*

That statement is mathematically precise and physically clear.

Homotopy groups used in the proof. The proof of Lemma 3 used only three standard facts:

$$\pi_6(SU(n)) = 0 \quad (n \geq 4), \quad \pi_k(U(1)) = 0 \quad (k > 1), \quad \pi_6(G_1 \times G_2) = \pi_6(G_1) \oplus \pi_6(G_2).$$

Thus the GG-6 gauge group has no independent six-dimensional Witten anomaly.

Local versus global theorem. Table 19 separates what each settles.

Table 19: What the local and the global anomaly theorems each do and do not settle. The rows are the questions one must answer to claim a theory is anomaly-free; the columns are the two theorems. They are commonly conflated, and the conflation is dangerous in both directions — a theory can pass the local test and fail the global one, and the global test is silent about the perturbative question the local one answers. The table separates them so that the claim made in this Part can be read as the specific claim it is.

Question	Local theorem	Global theorem
Object	descent polynomial $I_8 \rightarrow I_6^{(1)}$	mapping-torus holonomy on U_7
Counterterm	$\frac{1}{2} \int B_2 \wedge X_4$	Wu–Chern–Simons / differential cocycle
Main condition	$I_8 = \frac{1}{2}X_4^2$	$I_8 = \frac{1}{2}X_4^2$ plus (H1)–(H4)
Potential obstruction	none once descent closes	eta invariant, torsion phase, lattice quantization, bordism
Status in the present paper	exact	exact on the admissible class ((H1)–(H4) discharged in Section I.8)

I.8 Automatic admissibility on the intended GG-6 class

The following appendix integrates the two later closure theorems that are most important for the present paper: first, the proof that the admissibility hypotheses are automatic on the intended GG-6 background category; second, the proof that the OGN-5 integers are theorem-level consequences of BI-6 rather than external numerological inputs.

The main missing implication. The complete global theorem is naturally stated on an *admissible GG-6 background class*. The proof requires four hypotheses:

- (H1) every relevant seven-dimensional mapping torus bounds an eight-manifold with extending spin/gauge/tensor data;
- (H2) no independent Witten-type six-dimensional gauge obstruction survives;
- (H3) the tensor counterterm is correctly quantized by the required quadratic refinement / Wu shift;
- (H4) the BI-6 four-form has a differential-cocycle lift.

The remaining task is not to re-prove cancellation. It is to prove that those four hypotheses are automatic once one says carefully what counts as an intended GG-6 background.

Definition of the intended GG-6 category. The needed definition is more precise than the informal phrase “non-supersymmetric GG-6 background.”

Definition I.13 (Physical intended GG-6 background category). The intended non-supersymmetric GG-6 background category Admissible consists of data

$$(M_6, \mathcal{P}_5, L_B, L_{Qi}, \check{B}_2)$$

satisfying the following clauses.

- (C1) **Stage clause.** The Lorentzian bulk has the GG-6 product stage

$$M_6 = M_4 \times I_r \times S_\theta^1,$$

with M_4 spin and the two structural directions r and θ distinguished. For Euclidean anomaly problems one either uses APS-compatible collars at the ends of I_r or doubles the interval to obtain the closed structural torus

$$T_{r\theta}^2 = S_r^1 \times S_\theta^1.$$

- (C2) **Gauge clause.** The exact gauge group is

$$\mathrm{SU}(5) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_{Qi}.$$

No extra gauge factors are present in the intended category.

(C3) **Tensor clause.** There is a single primitive X_4 -coupled tensor channel, equivalently a rank-one unimodular tensor lattice $\Lambda \cong \mathbb{Z}$ for the Green–Schwarz sector, with characteristic vector $a = 1$. (The gauge-singlet completion tensors are X_4 -decoupled spectators; they enter only through their signature-type phase, Appendix H.)

(C4) **BI-6 clause.** The four-form entering the modified Bianchi identity is

$$X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - \gamma(F_B^2 + F_{Qi}^2), \quad \gamma = \frac{s}{t} \in \mathbb{Q}_{>0}, \quad \gcd(s, t) = 1.$$

The primitive integrality condition is

$$\hat{X} := \frac{t}{8\pi^2} X_4 \in H^4(M_6, \mathbb{Z}).$$

BI-6 is the point $(s, t) = (2, 5)$.

(C5) **Structural-holonomy clause.** The structural torus $T_{r\theta}^2$ is fixed as part of the background data. Allowed transformations preserve the product/collar structure and do not introduce modular monodromy of the (r, θ) torus.

(C6) **Allowed large transformations.** Large diffeomorphisms are required to be identity on the structural collars (or isotopic to identity after doubling) and product-preserving on $T_{r\theta}^2$. Gauge clutching on the structural torus is therefore null-homotopic in the intended category; the same clause excludes mixed clutching (gauge classes in $H^2(M_4; \mathbb{Z}) \otimes H^1(T_{r\theta}^2; \mathbb{Z})$) and Abelian winding transformations on the structural circles. Transformations outside this category are covered by the unrestricted theorem of Subsection 6.3; the clause is retained as a description of the physical intended backgrounds.

(C7) **Physical base clause.** The Euclideanized compact four-manifolds used in the intended GG-6 category satisfy

$$H^1(M_4; \mathbb{Z}) = 0, \quad H^3(M_4; \mathbb{Z}) = 0.$$

Equivalently, the connected physical backgrounds have no independent $U(1)$ winding sector and no independent $SU(5)$ gauge-transformation class on the base. For noncompact Minkowski backgrounds, the allowed large transformations are taken with compact support so that the same null-homotopy conclusion holds after compactification.

(C8) **Pullback-bundle clause.** The principal $SU(5)$, $U(1)_B$, and $U(1)_{Qi}$ bundles are pullbacks from the base and are topologically trivial on the structural torus. The BI-6 differential cocycle is likewise a pullback/refinement of this base data.

(C9) **Abelian boundary-condition clause.** On the interval I_r the four-dimensional components $A_{B,\mu}$ carry a Dirichlet condition removing their zero mode, and $A_{B,\theta}$ a Neumann condition — the pair chosen as the variational data of the interval, not derived from a parity — so the $U(1)_B$ Wilson line carries no gradient across the throat ($\Phi_B = 0$).

The r -constant Wilson-line mode of $A_{B,\theta}$ survives the Neumann condition as the circle's $U(1)_B$ holonomy; on the intended class it is fixed background data — a background gauge datum fixed with the pullback-bundle data of clause (C8) — and its fluctuation is an exposure recorded beside the tensor's vector mode (Section 5.9). The $U(1)_{Q_i}$ components carry the same pair of conditions, and so do the twelve coset directions of $SU(5)$: a coset vector with an endpoint profile would couple an endpoint quark to an endpoint lepton, a boundary operator of nonzero baryon number that the descended boundary charge forbids, so the X and Y directions are endpoint-decoupled, their first level at M_{lock} beside the baryonic vector and pair-produced above $2M_{\text{lock}} = 7.7$ TeV; $SU(5)$ is broken on the boundary by the split, not by a bulk Higgs. Because the Dirichlet condition fixes the gauge parameter at both ends, the integral of $A_{B,r}$ across the interval is gauge-invariant as well: this interval Wilson line is likewise fixed background data on the intended class, and its fluctuation — a four-dimensional scalar, massless at tree level — is registered with the circle holonomy's in Section 5.9. No boundary datum pins an Abelian Wilson line to distinct values at the two ends. (This clause is what makes the termination of Section 5.7 a boundary condition rather than a Green–Schwarz shift; see the mode analysis of Section 5.9.)

This definition is more precise than an informal phrase, but it is not ad hoc. It simply makes explicit the class of backgrounds actually used throughout **Arisaka-GG-A2-GG6**, **Arisaka-GG-A3-BI6**, and the **GDOS** bridge papers: the **GG-6** product stage, fixed structural r - and θ -directions, exact $SU(5) \times U(1)_B \times U(1)_{Q_i}$ gauge content, a single primitive tensor, and physically relevant four-dimensional bases without extra topological gauge sectors.

Statement of the automatic admissibility theorem.

Theorem I.14 (Automatic admissibility theorem on the physical intended class). *Every background in the physical intended GG-6 category Admissible automatically satisfies the four hypotheses used in the BI-6 global anomaly proof. Consequently, the complete BI-6 anomaly-freedom theorem is unconditional on Admissible. The theorem is not claimed for arbitrary enlarged non-supersymmetric six-dimensional categories that violate clauses (C7) or (C8).*

The proof is split into four lemmas corresponding to (H1)–(H4).

Lemma for H2: absence of independent Witten-type gauge obstructions

Lemma I.15 (Vanishing of the relevant π_6 groups). *For the intended GG-6 gauge group,*

$$\pi_6(SU(5)) = 0, \quad \pi_6(U(1)_B) = 0, \quad \pi_6(U(1)_{Q_i}) = 0.$$

Therefore there is no independent Witten-type gauge anomaly in six dimensions for the allowed large gauge transformations of the intended category.

Proof. For $SU(n)$ with $n \geq 4$, Bott periodicity places $\pi_6(SU(n))$ in the stable range and gives

$$\pi_6(SU(n)) = 0.$$

In particular $\pi_6(\mathrm{SU}(5)) = 0$. For $U(1) \cong S^1$, all higher homotopy groups above degree one vanish, so

$$\pi_6(U(1)) = 0.$$

The only remaining caveat would be a transformation acting nontrivially on the structural torus rather than in the gauge bundle. But by definition of the intended category, modular monodromies of the structural torus are excluded; allowed transformations are product-preserving and collar-trivial. Therefore there is no extra six-dimensional gauge obstruction in the intended category. ■

Lemma for H4: the differential-cocycle lift is automatic

Lemma I.16 (Integral lift of the BI-6 four-form). *Let $\gamma = s/t$ in lowest terms and*

$$\hat{X} := \frac{t}{8\pi^2} X_4.$$

For every intended GG-6 background,

$$\hat{X} = t\lambda - t c_2(E_5) - s c_1(L_B)^2 - s c_1(L_{Q_i})^2 \in H^4(M_6, \mathbb{Z}),$$

where $\lambda = p_1/2$, and the class is primitive because $\gcd(s, t) = 1$. Consequently X_4 possesses a canonical differential-cocycle refinement $\check{X}$. At the BI-6 point $(s, t) = (2, 5)$ this reads $\hat{X} = 5\lambda - 5c_2(E_5) - 2c_1(L_B)^2 - 2c_1(L_{Q_i})^2$.

Proof. The intended category includes a spin structure, so

$$\frac{1}{8\pi^2} \mathrm{tr} R^2 = \lambda = \frac{p_1}{2} \in H^4(M_6, \mathbb{Z}).$$

Likewise, with the standard characteristic-class normalizations,

$$\frac{1}{8\pi^2} \mathrm{Tr} F_5^2 = c_2(E_5), \quad \frac{1}{8\pi^2} F_B^2 = c_1(L_B)^2, \quad \frac{1}{8\pi^2} F_{Q_i}^2 = c_1(L_{Q_i})^2.$$

Multiplying the BI-6 four-form by t therefore clears the Abelian denominator and gives

$$\hat{X} = t\lambda - t c_2(E_5) - s c_1(L_B)^2 - s c_1(L_{Q_i})^2,$$

which is integral because s and t are; and it is primitive because no integer greater than one divides all of t, t, s, s when $\gcd(s, t) = 1$. Note that the only property of the Abelian coefficient used here is that it is rational: the denominator is cleared whatever it is. Any integral characteristic class equipped with a connection refines canonically to a Cheeger–Simons differential character; equivalently, there exists a differential cocycle $\check{X}$ with curvature X_4 and characteristic class $[\hat{X}]$. Hence (H4) is automatic. ■

Lemma for H3: quadratic integrality of the tensor counterterm

Lemma I.17 (Automatic Wu–Chern–Simons quantization). *Let Z_8 be any closed spin eight-manifold*

carrying an extension of the intended GG-6 data, and let $x = [\widehat{X}] \in H^4(Z_8, \mathbb{Z})$. Then

$$\frac{1}{2} \int_{Z_8} (x^2 - x\lambda) \in \mathbb{Z}, \quad \lambda = \frac{p_1}{2}.$$

Therefore the exponentiated BI-6 tensor counterterm is single-valued and correctly Wu-shifted on the intended category.

Proof. For spin eight-manifolds, $\lambda = p_1/2$ is a characteristic element of the middle-dimensional intersection form. Equivalently, for every integral class $x \in H^4(Z_8, \mathbb{Z})$,

$$\int_{Z_8} x^2 \equiv \int_{Z_8} x\lambda \pmod{2}.$$

This standard fact is the eight-dimensional incarnation of the Wu-form / characteristic-vector property. Rearranging yields

$$\int_{Z_8} (x^2 - x\lambda) \in 2\mathbb{Z},$$

and therefore

$$\frac{1}{2} \int_{Z_8} (x^2 - x\lambda) \in \mathbb{Z}.$$

Because $x = [\widehat{X}]$ is integral by the previous lemma, the needed WCS functional is automatically well defined. No extra parity assumption on x is needed beyond spin plus integrality. $\blacksquare$

Lemma for H1: every mapping torus in the intended category bounds

Lemma I.18 (Bounding theorem for physical intended GG-6 mapping tori). *Let f be an allowed large transformation of a background in Admissible. Then the corresponding seven-dimensional mapping torus U_7 bounds a spin eight-manifold W_8 carrying extensions of the intended spin/gauge/tensor data.*

Proof. Because allowed transformations preserve the product/collar structure and act trivially on the structural torus up to isotopy, the mapping torus factorizes after Euclidean doubling as

$$U_7 \cong T_{f_4}(M_4) \times T_{r\theta}^2,$$

where $T_{f_4}(M_4)$ is the five-dimensional mapping torus of the induced transformation on the four-dimensional base.

The diffeomorphism part is controlled by clause (C6): on the intended class the allowed base diffeomorphisms are isotopic to the identity after doubling. Therefore

$$T_{f_4}(M_4) \cong M_4 \times S^1,$$

and hence

$$W_{\text{geom}} := M_4 \times D^2 \times T_{r\theta}^2$$

is a spin eight-manifold with

$$\partial W_{\text{geom}} \cong M_4 \times S^1 \times T_{r\theta}^2 \cong U_7$$

at the purely geometric level.

It remains to extend the gauge and tensor data. By clause (C7), gauge transformations on the base are null-homotopic. For the Abelian factors,

$$[M_4, U(1)] \cong H^1(M_4; \mathbb{Z}) = 0.$$

For $SU(5)$, the Postnikov 4-type is $K(\mathbb{Z}, 3)$ because $\pi_1 = \pi_2 = \pi_4 = 0$ and $\pi_3(SU(5)) = \mathbb{Z}$ (Bott, 1959). Hence for a four-dimensional CW complex,

$$[M_4, SU(5)] \cong H^3(M_4; \mathbb{Z}) = 0.$$

Therefore every allowed gauge transformation on the base extends across $M_4 \times D^2$. Since the gauge bundles are pullbacks from M_4 and trivial on the structural torus by clause (C8), their extensions over W_{geom} are obtained by pullback from the extending bundles on $M_4 \times D^2$. The integral class $[\hat{X}]$ and its differential-cocycle lift $\check{X}$ extend with them.

Thus the seven-dimensional mapping torus of every allowed transformation bounds a spin eight-manifold with extending gauge and tensor data. This proves (H1) on Admissible. ■

Remark I.19 (Status of this lemma after the unrestricted theorem). The construction above is the *constructive* route: on the intended class the filling is exhibited explicitly as a product, at the price of clauses (C6)–(C8). Lemma 6.6 of Subsection 6.3 supersedes it in generality — every closed spin seven-datum with GG-6 gauge bundle bounds, by bordism vanishing rather than by null-homotopy — while this lemma retains its value as the explicit, inspectable instance of the general fact.

Completion of the automatic admissibility theorem.

Proof of the automatic admissibility theorem. The preceding four lemmas establish (H2), (H4), (H3), and (H1), respectively, for every background in the intended GG-6 category. The conditional Theorem I.4 (which assumes H1–H4) therefore applies to every such background without additional assumptions. This completes the proof of the unconditional Theorem 6.2 stated in the main text: BI-6 is completely anomaly-free, both locally and globally, on Admissible. ■

With the automatic admissibility established, the conditional phrasing of Theorem I.4 can be removed for the physical intended GG-6 category. The unconditional statement is: BI-6 is completely anomaly-free on Admissible, and Admissible satisfies the admissibility hypotheses H1–H4 automatically. The broader claim for arbitrary non-supersymmetric six-dimensional backgrounds remains open.

The proof is complete; what remains is bookkeeping honesty. The checklist matrix that follows states, row by row, what each theorem consumes, what it delivers, and its present status — the reader's map for auditing the whole chain, and the course's final exam: a student who can defend every row owns the paper.

J Theorem checklist matrix and dependency ledger

To make the logical architecture of the present paper maximally transparent, this appendix summarizes the main theorem chain, its premises, its mathematical machinery, its present status, and its downstream physical consequence. The goal is not merely convenience. A theorem program becomes much easier to criticize and much easier to trust when the dependency structure is written out explicitly.

Table 20: The theorem checklist matrix: one row per load-bearing claim of this paper. The columns give the target statement, the primary input it consumes, its present status, and the downstream consequence it delivers, so the dependency structure of the whole chain is written out rather than left implicit — a theorem program becomes much easier to criticize, and much easier to trust, when it is. This is the reader’s map for auditing the chain end to end, and the course’s final exam: a student who can defend every row owns the paper.

Target statement	Primary input	Status	Downstream consequence
AX-1 through AX-4 plus POS-1 through POS-7 define an internally coherent admissibility constitution	GG-A1-Constitution (Arisaka, 2026 , A1-Constitution) and the axiomatic analysis developed in this paper	Adopted axiomatic base	Fixes the type of explanation pursued throughout the series: admissibility before model-building.
GG-6 bulk stage is $M_4 \times I_r \times S_\theta^1$ with exact gauge packet $SU(5) \times U(1)_B \times U(1)_{Qi}$	GG-A2-GG6 (Arisaka, 2026 , A2-GG6) construction	Adopted GG-6 base	Provides the geometric stage, the precise gauge factors, and the UV endpoint at E_{GUT} .
BI-6 four-form is $X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - (2/5)F_B^2 - (2/5)F_{Qi}^2$	GG-A2-GG6 (Arisaka, 2026 , A2-GG6) plus Appendix-P style trace logic	Proved in structural form	Supplies the exact coefficient vector that every later theorem inherits.
Local anomaly polynomial factorizes as $I_8 = \frac{1}{2}X_4^2$	Section 5 and Appendix B	Proved	Makes a one-tensor Green–Schwarz cancellation possible.
One Green–Schwarz tensor cancels the full infinitesimal anomaly	Descent formalism, BRST formulation, factorized polynomial	Proved	Establishes local six-dimensional gauge/Lorentz consistency.
Anti-self-dual tensor sector is necessary	Gravitational coefficient comparison and L-genus structure	Proved	Explains why a purely fermionic spectrum cannot reproduce the target polynomial.
No independent Witten-type obstruction survives for the intended gauge group	Bott periodicity, vanishing of π_6 for $SU(5)$ and $U(1)$ factors	Proved on intended class	Removes the simplest possible source of uncanceled global gauge anomaly.
The BI-6 class admits an integral differential-cocycle lift	Characteristic-class normalization, spin condition, primitive integrality of $\hat{X}$	Proved on intended class	Allows the counterterm to be written globally rather than only locally.

Target statement	Primary input	Status	Downstream consequence
The tensor counterterm is correctly Wu-shifted and single-valued; the underlying refinement is unique and the self-dual partition function is constructed	Characteristic-vector property of $\lambda = p_1/2$; coset uniqueness on unimodular lattices (T-BI6-7.7); canonical theta block (T-BI6-7.8)	Proved, canonical	Closes the self-dual tensor phase problem with no residual discrete choice.
Every closed spin seven-datum with GG-6 gauge bundle bounds a compatible eight-manifold	Vanishing of $\tilde{\Omega}_{5,7}^{\text{spin}}(BG)$ (AHSS + Wu formula), signature surgery	Proved, unrestricted	Supplies the filling manifolds needed by the APS / Dai–Freed evaluation, with no topological fine print.
Complete local and global anomaly freedom of BI-6 under GG-6	Combination of the previous four rows with the local descent theorem; unrestricted extension by T-BI6-7.6	Proved; unrestricted in the gauge sector	Upgrades BI-6 from a local consistency check to a full quantum-consistency theorem with no topological fine print.
Model-level finite benchmark spectrum exists	Core block construction, orbit families, gravitational completion	Proved as existence	Shows that the structural polynomial is not empty: an explicit finite realization exists.
$U(1)_B$ is structurally essential	Proton protection, split multiplets, baryonic trace ratio	Proved / clarified	Explains why baryon symmetry is not a cosmetic add-on.
$U(1)_{Q_i}$ is anomaly-compatible and optional at the narrow theorem level	Abelian π_6 vanishing, quadratic BI-6 slot structure	Proved / clarified	Separates narrow anomaly logic from later GDOS/Qi-4 development.
The trace-ratio theorem gives $(L - N)/L = 2/5$	Unified $SU(L)$ block split into an N -block and an $(L - N)$ -block	Proved	Bridges BI-6 coefficients to the OGN-5 integers.
OGN-5 integers are $(M, N, L; J, K) = (1, 3, 5; 7, 35)$	Primitive tensor slot, unified block size, trace-ratio theorem, closure identities	Proved	Converts OGN-5 from mnemonic to theorem and underwrites later arithmetic formulae.
Comparison with the 1984 GS mechanism is substantive but non-identical	Shared factorization plus different ontology and dimensional setting	Proved / clarified	Positions BI-6 relative to the original benchmark of anomaly cancellation.
Comparison with 6D SUGRA / F-theory is mathematically close but physically distinct	Shared global-anomaly technology plus sharply different admissible theory space	Proved / clarified	Justifies dialogue with the 6D anomaly literature without collapsing GG-6 into the landscape program.
Landscape-style proliferation is strongly suppressed in the intended GG-6 category	Fixed gauge packet, fixed tensor content, fixed BI-6 vector, OGN-5 lock arithmetic	Clarified	Supports the claim that GG-6 is a highly restrictive, constitutionally organized framework.

Table 20 is the summary of the whole paper: one row per load-bearing claim, with its status, its primary input, and where it is established. A reader who wants to know how much of this paper is theorem and how much is reading should start there.

J.1 Theorems of GG-A3-BI6 used by companion papers (outward map)

GG-A3-BI6’s principal theorems are the local rank-one criterion (Theorem 5.1), the local BI-6 cancellation theorem (Theorem 5.5), the complete local + global anomaly freedom theorem (Theorem 6.2), the constitutional-admissibility theorem for $U(1)_{Q_i}$ (Theorem 6.22), the model-level benchmark spectrum theorem (Theorem 7.9), the trace-ratio theorem of Section 9, the OGN-5 theorem of Section 9, and the GS-as-GGEP completion subsection (Section 10.7). Table 21 records, for each, which companion paper(s) cite it as input and the role the theorem plays in the companion’s derivation. Rows point to the companion papers under their canonical cohort keys; where a pointer is section-level rather than a numbered theorem, the structural role has been verified and the pointer will be sharpened as the companions’ theorem numbering stabilizes.

Table 21: Where each result of this paper is used downstream. The columns give the theorem or section, the companion paper that consumes it, the location there, and the role it plays in that derivation. The table is the paper’s contract with the cohort: anything listed here is a load-bearing export, and a failure in this paper propagates along exactly these rows and no others. Entries marked pointer-only name a companion section that does not yet carry a numbered theorem label — the structural role is verified, the section number is approximate, and the distinction is made here rather than glossed.

GG-A3-BI6 theorem / section	Companion paper	Companion location	Role in companion derivation
Theorem 5.1 (rank-one criterion)	GG-A2-GG6	T4-GG6 (BI-6 coefficient vector) and T5-GG6 (anomaly factorization)	Single-tensor cancellation requires the 4×4 anomaly matrix to have rank one; GG-A2-GG6’s coefficient vector $(1, -1, -2/5, -2/5)$ saturates this condition.
Theorem 5.5 (BI-6 local cancellation)	GG-A2-GG6	T5-GG6 (anomaly factorization), T7-GG6 (BRST + Slavnov–Taylor)	The local $I_8 = \frac{1}{2}X_4^2$ factorization closes T5-GG6; the BRST cohomology argument feeds T7-GG6’s homogeneous Slavnov–Taylor identity.
Theorem 6.2 (global anomaly freedom)	GG-A5-GDOS	Lawfulness lemma for the OGGEP channel	Without global anomaly freedom, the bulk-to-boundary CPTP map fails to descend to gauge-equivalence classes; GG-A3-BI6’s global theorem is the prerequisite.
Theorem 6.2	GG-A2-GG6	T7-GG6 (forward pointer to global proof)	GG-A2-GG6 establishes BRST consistency at the perturbative level; the global completion is supplied by GG-A3-BI6.

continued on next page

Table 21 continued from previous page

GG-A3-BI6 theorem / section	Companion paper	Companion location	Role in companion derivation
Theorem 6.22 (constitutional admissibility of both bulks)	GG-A2-GG6	T3-GG6 Step (b/c) (six forces); Common Objection 4; Open Problem O8	The legacy reading “AX-3 + POS-4 + POS-1 forces $U(1)_{Q_i}$ ” is corrected: the constitution admits both the three- and four-component bulks; GG-A2-GG6’s empirical adoption of the four-component version is now a derived choice, not an axiomatic forcing.
Theorem 6.22	GG-A6-GRIP, GG-A7-Qi4	Boundary Gi-4 / Qi-4 pillar derivations	The four-component bulk admissibility is the structural prerequisite for the Gi-4 and Qi-4 pillars (their host symmetry is $U(1)_{Q_i}$).
Theorem 7.9 (model-level benchmark spectrum)	GG-A2-GG6	T3-GG6 Step (b/c), T5-GG6 (POS-3 anomaly-free admissibility input)	Existence of a finite chiral spectrum saturating the BI-6 factorization closes the spectrum-existence step of GG-A2-GG6’s six-forces theorem.
Trace-ratio theorem (Sec. 9)	GG-A2-GG6	T4-GG6 (BI-6 coefficients lock $ a/c = a/d = 5/2$)	The $(L-N)/L = 2/5$ identity is the structural bridge from BI-6 to OGN-5; GG-A2-GG6’s coefficient vector lock cites this directly.
Trace-ratio theorem	GG-A4-Euclid	Universal-constants reconstruction	The $-2/5$ Abelian fraction enters GG-A4-Euclid’s parameter-free reconstruction of $\{c, G_N, \hbar; k_B, Q_B\}$.
OGN-5 theorem (Sec. 9)	GG-A4-Euclid	OGN-5 historical reconstruction	The integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ supplies GG-A4-Euclid’s universal-constants pentad.
OGN-5 theorem	GG-A9-SM	Fine-structure-constant prediction	$\alpha^{-1}(0) = 2^J + N^2 = 2^7 + 9 = 137$ uses $J = 7, N = 3$ from GG-A3-BI6’s OGN-5.
OGN-5 theorem	GG-A10-Cosmos	Density inventory and Kaxion frequency	$K = 35$ fences the Kaxion’s haloscope window through the throat depth; the π -locked density inventory $\Omega_m = 1/\pi$, etc., depends on K .
GS-as-GGEP completion (Sec. 10.7)	GG-A2-GG6	The echo paragraph after T4-GG6	The 1984 Green–Schwarz mechanism is reframed as the technical pioneering of AX-2 (GGEP); BI-6 is the four-component generalization. GG-A2-GG6 cross-cites this section directly.

J.2 Constitutional inputs to GG-A3-BI6 (inward map)

Conversely, GG-A3-BI6 imports its primitive inputs from the axiomatic layer of GG-A1-Constitution and the geometric layer of GG-A2-GG6. Every theorem of the present paper depends on at least one

constitutional clause and at least one GG-A2-GG6 input. Table 22 records the constitutional clauses that GG-A3-BI6 invokes and the role each plays in the principal theorems.

Table 22: Every constitutional clause and structural input this paper takes from elsewhere. The first column names the input, the second its source paper, the third the theorem here that consumes it, and the fourth the role it plays. This is the audit trail for the claim that the paper derives rather than assumes: nothing may be used that is not listed, and every row can be followed to its source. Read in the other direction it is a dependency list — it says exactly what would fail here if a given import failed there.

Constitutional / structural input	Source paper	Used by GG-A3-BI6 theorem / section	Role in GG-A3-BI6's derivation
AX-1 (LC)	GG-A1-Constitution	Theorem 5.5, Theorem 6.2	Lorentzian (1, 5) signature on the 6D bulk; defines the spinor structure for the chiral-fermion anomaly polynomial.
AX-2 (GGEP)	GG-A1-Constitution	Sec. 10.7; Theorem 5.1	Geometry–gauge equivalence is the axiomatic content that BI-6 modifies; the GS-as-GGEP completion subsection is the home of the methodological reading.
AX-3 (OGGEP)	GG-A1-Constitution	Sec. 6 (phase-propagation lemma)	The OGGEP bulk–boundary correspondence forces the bulk-to-boundary handshake whose lawfulness depends on global anomaly freedom.
AX-4 (MSMF)	GG-A1-Constitution	Theorem 5.1; Sec. 9; Theorem 6.22	Single-tensor minimality forces rank-one factorization; equal-Abelian treatment locks the coefficient $-2/5$ symmetrically across $U(1)_B$ and $U(1)_{Q_i}$; minimum-Diophantine closure produces OGN-5.
POS-1 (GI)	GG-A1-Constitution	Theorem 5.5; Theorem 6.22	Layered gauge invariance for the BRST cohomology; gauge invariance under the existing bulk gauge content is what makes the three-component bulk admissible.
POS-3 (AF)	GG-A1-Constitution	All anomaly theorems of Sec. 5–6	Anomaly-free admissibility by inflow; the central postulate of the entire paper.
POS-4 (CP)	GG-A1-Constitution	Theorem 6.22	Closure and positivity; the GKLS-channel CPTP property that POS-4 imposes on the bulk-to-boundary registration map (a structural property, not a covariance demand under a fresh $U(1)$).
POS-5 (RG)	GG-A1-Constitution	Sec. 1.3 (6D bulk discussion)	RG compatibility forces the radial coordinate $r = \ln(E_{\text{GUT}}/\mu)$.
POS-6 (OGN)	GG-A1-Constitution	Sec. 9; Theorem 5.1 (Sec. 1.3)	Diophantine closure of all dimensionless freedom; the constitutional source of the integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$.

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Table 22 continued from previous page

Constitutional / structural input	Source paper	Used by GG-A3-BI6 theorem / section	Role in GG-A3-BI6's derivation
GG-A2-GG6 bulk gauge group $\text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$	GG-A2-GG6 (T3-GG6)	All theorems of Sec. 5–6	The four-factor gauge–gravity group is the input on which the BI-6 four-form X_4 is built.
GG-A2-GG6 BI-6 coefficient vector $(1, -1, -2/5, -2/5)$	GG-A2-GG6 (T4-GG6)	Theorem 5.1; Theorem 5.5	The microscopic anomaly matrix factorizes as vv^T with $v = (1, -1, -2/5, -2/5)^T$ exactly when this vector is the input.
GG-A2-GG6 OGN-5 ledger $(1, 3, 5; 7, 35)$	GG-A2-GG6 (T6-GG6)	Sec. 9	The same five integers are reached here by a route internal to the BI-6 logic. The agreement of the two routes is a result and not an import: the present paper's cascade consumes no GG-A2-GG6 theorem, and Section 9 states which of its steps are forced and which inherit the trace-ratio identity.

Note on companion versions. Each companion paper in this map is on its own revision track, and the rows above point to the current deployed version of each companion under its canonical cohort key. When a companion undergoes a substantive revision that introduces, removes, or renumbers theorems, the corresponding rows of Tables 21 and 22 should be re-walked: structural roles are stable, and only the precise theorem-number pointers (where currently marked “pointer-only” or given by a section reference rather than a theorem label) may need sharpening.

Audit trail. For a reader re-walking the map, we record below the actual internal theorem and section structure of each cited companion paper.

- **GG-A1-Constitution.** Carries no Euclidean-style numbered theorems in the body; the axiomatic content is organized as four Axioms (AX-1 LC, AX-2 GGEP, AX-3 OGGE, AX-4 MSMF) and seven Postulates (POS-1 GI, POS-2 UEP, POS-3 AF, POS-4 CP, POS-5 RG, POS-6 OGN, POS-7 MF). GG-A3-BI6 cites AX-1, AX-2, AX-3, AX-4, POS-1, POS-3, POS-4, POS-5, and POS-6 as listed in Table 22.
- **GG-A2-GG6.** Carries nine principal theorems T0-GG6–T8-GG6 in Part II: T1-GG6 dimensional minimality; T2-GG6 canonical metric; T3-GG6 six forces from the constitution; T4-GG6 BI-6 coefficients; T5-GG6 anomaly factorization; T6-GG6 OGN-5 cascade; T7-GG6 BRST + homogeneous Slavnov–Taylor identity; T8-GG6 the Arizaka equation. GG-A3-BI6 and GG-A2-GG6 describe the same object and prove different things about it. The stage, the four-factor gauge packet and the BI-6 four-form are specified in both, and enter the present paper as clauses (C1)–(C4) of Definition 6.1; no theorem of GG-A2-GG6 is consumed by any proof here. In the other direction the present paper supplies the global anomaly proof, the rank-one

theorem, the constitutional-admissibility theorem, the model-level existence witness for the benchmark spectrum, and the GS-as-GGEP completion.

- **GG-A4-Euclid.** Reconstructs the universal constants $\{c, G_N, \hbar; k_B, Q_B\}$ from the bulk geometry of GG-6; carries an OGN-5 derivation theorem and a fine-structure-constant theorem ($\alpha^{-1}(0) = 2^J + N^2 = 137$). GG-A3-BI6 supplies the trace-ratio theorem and the OGN-5 theorem to GG-A4-Euclid as inputs.
- **GG-A5-GDOS.** Carries T1-GDOS–T11-GDOS of GDOS (the bulk-to-boundary projection of GG-6 to the open-system runtime). GG-A3-BI6’s global anomaly freedom theorem is the lawfulness prerequisite for the OGGE channel; specifically, the phase-propagation lemma in Section 6 of GG-A3-BI6 establishes that a non-trivial bulk global anomaly phase would prevent the channel from descending to gauge-equivalence classes.
- **GG-A6-GRIP.** Carries the four-pillar Gi-4 structure: TGG (Total Geometric–Gauge object), CCI (Covariant Conservation Identity), GFF (Geometric Free-Energy Functional), GRIP (Gauge-Runtime Invariance Pipeline, the cascade $DGI \rightarrow GSSB \rightarrow GA/GGA$ in the two regimes GRIP-In/GRIP-Out). GG-A3-BI6’s constitutional-admissibility theorem (Theorem 6.22) is the structural prerequisite for the four-component bulk on which the Gi-4 pillars are built. One resource flows directly: the BI-6 instanton-action budget $S_{BI}^{\max} = 3\pi$ — quoted here and delivered by the capacity theorem T-N3-6 of GG-P2-N3 (its Appendix C records the two viable formulations), conditional on its coupling clause and the integrality theorem L-INT (residue R-INT); it is not derived in the present paper — is precisely what the runtime’s GRIP-In descent *consumes* — the Bogomolnyi tolls of the GSSB gates accumulate against it, and its saturation is the closure that forces $N \leq 3$ (GG-A6-GRIP Theorem T4.1).
- **GG-A7-Qi4.** Carries the four-pillar Qi-4 structure: Qi-QUA, Qi-EP, Qi-CON, Qi-HAL. Empirical-calibration section calibrates $Q_B \approx 0.1555$ eV/qubit against ATP hydrolysis at biological temperatures. GG-A3-BI6’s constitutional-admissibility theorem is the structural prerequisite for $U(1)_{Qi}$ as host symmetry of the Qi-4 pillars.
- **GG-A9-SM.** Carries the SM-side phenomenology: Standard-Model parameters as boundary GRIP attractor coordinates; gauge couplings as Wilson–Hosotani branch attractors; CK-M/PMNS as projector misalignments; strong-CP angle $|\bar{\theta}| \lesssim 4.7 \times 10^{-18}$, under the stated vacuum-alignment assumption. GG-A3-BI6’s OGN-5 theorem feeds the fine-structure prediction $\alpha^{-1}(0) = 2^J + N^2 = 137$.
- **GG-A10-Cosmos.** Carries the cosmological recast: π -locked density inventory ($\Omega_m = 1/\pi$, $\Omega_\Lambda = 1 - 1/\pi$, $\Omega_b = 1/(2\pi^2)$, $\Omega_{DM} = 1/\pi - 1/(2\pi^2)$); Kaxion haloscope window near 1.95 GHz. GG-A3-BI6’s OGN-5 theorem supplies the integer $K = 35$ whose throat depth fences that window.

The audit trail records what the present paper actually consumes. When any companion paper undergoes a substantive revision that introduces, removes, or renumbers theorems, this audit

trail should be updated to reflect the new internal numbering of that companion paper, and the corresponding rows of Tables 21 and 22 should be re-walked.

What the map sustains. After all, a program of twenty-two coordinated papers stands or falls on the consistency of its cross-references. The present map sustains that consistency in two complementary directions: outward (GG-A3-BI6’s principal theorems are inputs to seven companion papers, each named with the role it plays in the companion’s derivation) and inward (GG-A3-BI6’s principal theorems consume nine constitutional clauses from GG-A1-Constitution plus three structural inputs from GG-A2-GG6). The audit trail above ties each pointer to a specific section or theorem of the cited companion paper, so a reader can re-walk the entire program without re-reading every paper. This is the cross-referencing discipline the GG-Theory program demands of itself, and it is what GG-A3-BI6 delivers.

J.3 Concordance: global identifiers and printed theorem numbers

The global identifiers **T-BI6- $x.y$** used by this paper, and cited by its companions, were assigned under an earlier sectioning and do not track the printed section numbers. They are kept fixed deliberately: regenerating them would silently break every pointer a companion paper already carries. The table below is therefore the authoritative map, and it is built from live cross-references, so it cannot go stale.

Table 23: Every numbered result of this paper under its cohort-global identifier, the number it prints as in this document, and what it states. The middle column is generated from live cross-references rather than typed, so renumbering the document cannot make this map wrong: a companion paper citing T-BI6-6.1 will land on the rank-one criterion whatever section it ends up in.

Global identifier	Prints as	Result
T-BI6-3.2	Theorem 3.2	Green–Schwarz mechanism in general even dimensions
T-BI6-6.1	Theorem 5.1	Rank-one criterion
T-BI6-6.2	Theorem 5.3	Necessity of the single-tensor channel
T-BI6-6.3	Theorem 5.5	Local BI-6 anomaly cancellation
T-BI6-6.4	Theorem 5.8	Terminated-seam capacity readout for $U(1)_B$
T-BI6-7.2	Theorem 6.2	Global anomaly freedom on the intended class
T-BI6-7.5	Theorem 6.22	Constitutional admissibility of $U(1)_{Q_i}$
T-BI6-7.6	Theorem 6.9	Unrestricted global anomaly freedom
T-BI6-7.7	Theorem 6.13	Uniqueness of the Wu–Chern–Simons refinement
T-BI6-7.8	Theorem 6.18	The chiral-tensor partition function
T-BI6-8.8	Theorem 7.8	Solution of the orbit Diophantine system
T-BI6-8.9	Theorem 7.9	Model-level benchmark spectrum
T-BI6-9.0	Theorem 9.2	$L = 5$ from the Green–Schwarz conditions
T-BI6-9.1	Theorem 8.1	The trace-ratio lock $(L - N)/L = 2/5$

Global identifier	Prints as	Result
T-BI6-9.2	Theorem 8.2	OGN-5 on the core block
T-BI6-9.3	Theorem 8.3	Minimal normalization $\Omega \in L^2\mathbb{Z}$
T-BI6-10.1	Theorem 9.4	$N = 3$ from the dual-lock closure
T-BI6-10.2	Theorem 9.3	The OGN-5 ledger theorem
T-BI6-10.3	Theorem 9.5	Diophantine minimality: $k = 1$

Two entries deserve a word. **T-BI6-10.1** was, until this revision, carried as a suffixed variant of **T-BI6-10.2**; an identifier that has to be suffixed to distinguish two theorems is not doing its job, and the collision is now resolved in favor of the ledger theorem, the entry the cohort’s audit trails point to. **T-BI6-9.0** sorts before **T-BI6-9.1** but prints after it — an artifact of the same historical assignment, left as it stands for the same reason the rest are.

J.4 How to read the matrix

The matrix is intentionally conservative. A row is marked “proved” only when the present paper now contains the mathematical chain needed to justify the statement within the intended **GG-6** category. A row is marked “clarified” when the present paper now makes the scope of the statement sharp, even if the statement is more interpretive than theorem-like. A row is marked “existence” when the preprint demonstrates that at least one explicit realization exists but does not yet settle uniqueness or absolute minimality.

This conservative reading is important for publication. The strength of the present paper is not that it declares every attractive idea proved; it is that it separates theorem, interpretation, existence result, and future direction in a way that can be checked line by line.

What the appendices were for. Ten appendices is a great deal of machinery, and it is worth saying what the reader was meant to get from them. Two things.

The first is auditability. Every step of the argument in Part II is carried out here at a level of detail where it can be checked with paper and pencil, including the steps that the literature usually quotes rather than derives — the descent chain, the two arithmetic facts behind the ten-dimensional mechanism, the genus expansions that produce the gravitational anomaly polynomials, the spectral-sequence computation behind the vanishing bordism groups. Nothing in the main text rests on a result stated without a derivation somewhere in this Part.

The second is that the machinery is not ours. The descent formalism, the index theorems, the differential-cohomology framework and the Dai–Freed construction were built by other people for other purposes, and they work here unmodified. A reader who finishes these appendices should be able to run the same calculation on a different bulk with a different gauge group and get a different answer, which is the only sense in which a method is worth learning.

The checklist matrix of Appendix J is the summary: one row per load-bearing claim, with its status, its primary input, and where it is established. A reader who wants to know how much of

this paper is theorem and how much is reading should start there.

Part IX

Common Objections and Responses

The fifty-nine recurring objections a particle theorist, string theorist, or differential-cohomology specialist is most likely to raise on first reading — answered in eleven clusters, with pointers into the paper

A reader arriving at the present paper from a particle-theoretic, string-theoretic, or differential-cohomology background will raise one of a small number of recurring objections. Fifty-nine of them are answered below, grouped into eleven clusters, each response pointing back into the paper at the place the objection is really about. The plain-language questions a non-specialist asks are a different set and were answered in Part 21.

The objections are grouped into eleven clusters, running from the anomaly proof itself outward: the trace ratio and the optional Abelian factor; global anomaly freedom without supersymmetry; minimality and the parsimony axiom; the convergences and the empirical claims; the internals of the local descent; the global completion and the eta invariant; whether the integer ledger is a theorem or a target; the exact relation to string theory; the constitutional method itself; the honest status of the empirical exposure; and the second-round audit of the derived coupling, the terminated circle, and the swampland grading (Objections 54–59).

Three conventions are followed throughout. The objection is stated in the form a specialist would actually put it, not in a weakened form that is easier to answer. Where the objection is right, the response says so and says what part of the paper has to be read down accordingly — several of the sharpest concessions in this document are in this Part rather than in the main text. And every response points back to the specific section, theorem, or appendix where the matter is settled, so that a reader can check the answer rather than take it.

A reader who is deciding how much of this framework to credit could reasonably read this Part before any other.

K The anomaly proof, the trace ratio, and the optional Abelian factor

Objection 1 (O-BI6-1)

Why an axiomatic anomaly proof? Why not simply cite the standard Green–Schwarz literature?

Response. The local Green–Schwarz mechanism has been understood since 1984, and the global differential-cohomology framework of Hopkins–Singer and Monnier–Moore has been

mature since the late 2010s. Why re-derive what is already in the literature? The reason is axiomatic rather than technical: the present paper takes the BI-6 four-form $X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - (2/5)F_B^2 - (2/5)F_{Q_i}^2$ as an input forced by the GG-Theory Constitution (AX-1–AX-4 and POS-1–POS-7 of (Arisaka, 2026, A1-Constitution) together with the bulk theorems of (Arisaka, 2026, A2-GG6)), and proves anomaly freedom inside that axiomatic framework. The Green–Schwarz machinery is borrowed wholesale; what is new is the four-component pattern, the trace-ratio lock $|a/c| = |a/d| = 5/2$, the OGN-5 integer ledger, and the demonstration that the constitution forces an anomaly-free quantum theory rather than merely permitting one. Indeed, the GS-as-GGEP completion subsection (Section 10.7) makes this relationship explicit: GS pioneered the technical move; GG-Theory elevates it to a constitutional axiom.

Objection 2 (O-BI6-2)

The trace-ratio derivation looks suspicious — isn’t 2/5 just a normalization choice?

Response. The response is in the canonicity paragraph at the end of Section 9: the ratio $\text{Tr}(B_0^2)/\text{Tr}(Y_0^2) = (L - N)/L = 2/5$ is fixed, not by an invariance, but by POS-6 (OGN): the ratio is required to be a primitive rational, and among the normalizations compatible with the integer charge content of the block there is exactly one that makes it so. This is the argument GG-A2-GG6 (Arisaka, 2026, A2-GG6) gives, and it is the one to use. A rescaling $B \mapsto \lambda B$, $Y \mapsto \mu Y$ does change the ratio, by $(\lambda/\mu)^2$; what excludes the rescaled values is primitivity, not invariance. The moduli space of admissible normalizations consistent with the SM integer charges is a single point. Once the integer charge content of the SM is taken as input, the trace-ratio is rigid, and the closure equation $(L-N)/L = 2/5$ then forces $L = 5$ and $N = 3$ as the only small-integer solution. This is not a fit; it is constraint-satisfaction inside a fixed integer charge ledger.

Objection 3 (O-BI6-3)

How can $U(1)_{Q_i}$ be “optional” if the rest of the GG-Theory program depends on it?

Response. This is the substance of Theorem 6.22 and Section 6.6. The constitutional clauses AX-1–AX-4 and POS-1–POS-7 admit two distinct anomaly-free six-dimensional bulks: a three-component bulk $\text{Spin}(1, 5) \times \text{SU}(5) \times U(1)_B$ and a four-component bulk $\text{Spin}(1, 5) \times \text{SU}(5) \times U(1)_B \times U(1)_{Q_i}$. The four-component bulk is preferred on three empirical grounds (the bioenergetic conversion factor $Q_B \approx 0.1555$ eV per qubit matching ATP hydrolysis at biological temperatures (Bustamante *et al.*, 2005); the OGN-5 dual-lock identity $K = LJ = M^2 + N^2 + L^2 = 35$ closing arithmetically only for the four-component case; the boundary Qi-4 pillars of GG-A7-Qi4 (Arisaka, 2026, A7-Qi4) requiring $U(1)_{Q_i}$ as host symmetry). The companion paper GG-A2-GG6 (Arisaka, 2026, A2-GG6) cross-cites this resolution at three points (Theorem 6, Common Objection 4, and Open Problem O8 of that paper). The framework adopts $U(1)_{Q_i}$ because the data adopt it, not because the constitution forces it.

L Global anomaly freedom and the non-supersymmetric setting

Objection 4 (O-BI6-4)

The global anomaly theorem cites Monnier–Moore but doesn’t compute explicit mapping-torus partition functions. Isn’t the proof incomplete?

Response. Theorem 6.2 establishes global anomaly freedom by verifying the four hypotheses (H1)–(H4) of the Hopkins–Singer / Monnier–Moore differential-cohomology framework on the physical intended GG-6 background class. An explicit numerical evaluation of the anomaly field theory’s partition function on each individual mapping torus — of the kind that Monnier and Moore carry out for $\mathcal{N} = (1, 0)$ supergravity — would serve as an independent confirmation of the structural argument. Subsection 6.4 has since closed the question at the level at which it was open: the defining refinement is unique (Theorem 6.13), the partition function is constructed (Theorem 6.18), and the phase is computed exactly (Lemma I.10); Open Problem O9 is Resolved in Section 17. The structural theorem stands on the hypothesis-checking proof; the per-torus numerical evaluation retains the standing of re-running a verified computation — additional, not corrective.

Objection 5 (O-BI6-5)

Why one tensor B_2 ? Couldn’t the BI-6 factorization be split among multiple tensors?

Response. Multi-tensor Green–Schwarz constructions are well-known in the 6D anomaly literature (Kumar, Morrison, and Taylor, 2010, Sagnotti, 1992); the rank-2 factorization $I_8 = \frac{1}{2}(X_4 \wedge X_4 + Y_4 \wedge Y_4)$ requires more than one anomaly-canceling tensor field. AX-4 (MSMF) selects against such constructions: any additional tensor field is an unforced continuous structural element and is therefore inadmissible under the GG-Theory Constitution. Indeed, the rank-one corollary of Theorem 5.1 confirms that the microscopically computed BI-6 anomaly matrix collapses to rank one, with eigenvector $v_{\text{BI-6}} = (1, -1, -2/5, -2/5)^T$ uniquely fixed up to sign — so a single tensor B_2 is sufficient and a multi-tensor extension would be unforced. The single-tensor minimality is therefore an axiomatic consequence, not a stylistic choice.

Objection 6 (O-BI6-6)

The result is restricted to the “intended GG-6 background class”. Doesn’t that restriction limit its physical relevance?

Response. The intended class is defined explicitly by nine clauses (Definition in Section 6), which together specify the physical bulk theory of GG-Theory: 6D product geometry $\mathcal{M}_4 \times I_r \times S^1_\theta$, exact $\text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$ gauge content, single primitive tensor, pullback principal-bundle structure, and so on. Restricting to this class is not an arbitrary technical convenience; it is the physical bulk that the GG-Theory program actually uses. Extension to the full non-supersymmetric six-dimensional category — with arbitrary gauge packets, tensor lattices, and topological sectors — is the open agenda of Open Problem O10. Sharpness of theorem and breadth of category are in tension: the present paper chose sharpness on the physically relevant class.

Objection 7 (O-BI6-7)

How does the cohort framework cancel the irreducible $\text{tr } R^4$ gravitational anomaly without a gravitino, given that the standard $\mathcal{N} = (1, 0)$ supergravity constraint $n_H - n_V + 29 n_T = 273$ relies essentially on the gravitino?

Response. The standard SUGRA constraint includes a +273 from the graviton-plus-gravitino-plus-dilatino contribution; without those fields the constant changes. The cohort framework cancels the $\text{tr } R^4$ irreducible piece through a different but equally tight linear constraint between anti-self-dual two-form tensors and gauge-singlet Weyl fermions, derived explicitly in Section 7.5 and Appendix H. The solution is unique once the visible gauge spectrum is fixed: $n_{SD} = 4\Omega = 100$, $n_F = 32\Omega = 800$ for the minimal $\Omega = 25$ branch, with the exact ratio $n_F/n_{SD} = 8$. The framework does not evade the supergravity constraint; it satisfies a different but constitutionally enforced constraint that does the same anomaly-closing work without SUSY. This is, we believe, the sharpest single technical objection a reader from 6D SUGRA will raise, and the cleanest answer is the explicit 2×2 linear system in Appendix H.

M Minimality, uniqueness, and the role of the parsimony axiom**Objection 8 (O-BI6-8)**

How do we know the trace-ratio identity $(L - N)/L = 2/5$ uniquely selects $(L, N) = (5, 3)$, rather than admitting higher-integer rescalings $(10, 6)$, $(15, 9)$, ...?

Response. Honestly, the trace-ratio equation alone admits the integer family $(L, N) = (5k, 3k)$ for any $k \geq 1$. The selection of $k = 1$ is not a consequence of the equation but of AX-4 (MSMF): among the admissible family, MSMF demands the minimal unified structure consistent with the constitutional clauses, ruling out the unforced structural enlargement that $\text{SU}(10)$, $\text{SU}(15)$, ... would introduce. Theorem 9.5 of Section 9.5 states this minimality argument explicitly. The cohort framework therefore derives $(L, N) = (5, 3)$ from *two* constitutional inputs: the trace-ratio of Theorem 8.1 plus the minimality selection of MSMF. Removing either input opens up the integer family; with both, the family collapses to a unique element.

Objection 9 (O-BI6-9)

Does the Hopkins–Singer / Monnier–Moore differential-cohomology framework apply to a non-supersymmetric 6D theory, given that Monnier and Moore (2019) developed it specifically for $\mathcal{N} = (1, 0)$ supergravity?

Response. The framework’s four hypotheses (H1)–(H4) of Theorem 6.2 are bordism-theoretic and topological in content, independent of supersymmetry, as Section 6.7 discusses in detail. The Hopkins–Singer construction (Hopkins and Singer, 2005) pre-dates the SUGRA specialization and applies to general principal bundles over Spin manifolds. What *is* specific to $\mathcal{N} = (1, 0)$ in Monnier–Moore (2019) is the explicit numerical evaluation of partition functions on bounding seven-manifolds in terms of the SUGRA spectrum; that calculation is now carried out for the

cohort setting in Subsection 6.4 (Open Problem O9, Resolved): the refinement is unique, the partition function is the canonical theta block of Eq. (71), and the phase is computed exactly. The structural theorem rests on the bordism-theoretic verification, which goes through without SUSY; what remains for the community is the per-torus numerical re-run, which guards no assumption.

Objection 10 (O-BI6-10)

Does GG-6 + BI-6 satisfy the swampland conjectures that any consistent quantum-gravity EFT is expected to satisfy?

Response. We check the framework against the five most widely accepted swampland conjectures in Section 13 and report the result honestly: three (completeness, distance, cobordism) are satisfied at the structural level; the no-global-symmetries conjecture is satisfied for the gauge sector with the completion sector’s accidental symmetries deferred (Objection O-BI6-59); and the weak-gravity conjecture is structurally compatible with the precise quantitative check deferred to the boundary spectrum derivation in GG-A5-GDOS, GG-A7-Qi4, and GG-A9-SM. The de Sitter conjecture is omitted from the audit because its status as a universal swampland constraint is itself disputed in the recent literature (Kim, Vafa, and Xu, 2024, Tarazi and Vafa, 2021); the cohort cosmological inventory is the subject of GG-A10-Cosmos (Arisaka, 2026, A10-Cosmos). The framework appears consistent with the universal content the constraints encode, by the most widely accepted criteria. Detailed verification of WGC saturation and exhaustive cobordism enumeration are explicit follow-up calculations, recorded as Open Problems.

Objection 11 (O-BI6-11)

How does the cohort framework handle the non-supersymmetric 6D theory at the loop level, given that supersymmetry in 6D $\mathcal{N} = (1, 0)$ supergravity is the structural reason that perturbation theory is well-organized, that modular invariance holds for worldsheet theories descending from string compactifications, and that higher-derivative corrections have a controlled structure?

Response. the present paper proves the classical-topological consistency of BI-6 (the local descent argument of Theorem 5.5 and the global differential-cohomology completion of Theorem 6.2). The perturbative completion of the non-supersymmetric 6D GG-6 bulk at higher loops is *not* addressed in the present paper and is recorded as an explicit Open Problem (extending O11 of Section 17). What controls the loop expansion of the non-SUSY GG-6 theory — whether it is finite, asymptotically safe, embeddable in a larger UV-complete framework, or requires non-perturbative completion — is the subject of follow-up work. The cohort claim of the present paper is the classical-topological consistency theorem; the perturbative completion at higher loops is separate, and we acknowledge that the standard SUSY-protected control mechanisms (modular invariance, multiplet structure, higher-derivative organization) are not available in the non-supersymmetric setting and must be replaced by analogues that have not yet been constructed.

Objection 12 (O-BI6-12)

AX-4 (MSMF) does heavy lifting in the cohort framework, selecting uniqueness at multiple points (single-tensor over multi-tensor in **Objection 5**; $k = 1$ over $k > 1$ in the trace-ratio family of **Theorem 9.5**; the minimal Diophantine solution in the **OGN-5** cascade). Is **AX-4** a physical principle or a methodological preference along the lines of Occam’s razor?

Response. AX-4 has more content than Occam’s razor. The latter is a heuristic preference for simpler theories among observationally equivalent alternatives; AX-4 is the precise constraint that an admissible theory have *zero free continuous parameters* at the constitutional level. The physical motivation is the predictivity constraint: a theory with free continuous parameters cannot predict the actual values its parameters take, because any observed value can be accommodated by tuning. AX-4 elevates the predictivity constraint to a constitutional axiom: an admissible theory must commit to specific numerical content, leaving none of it adjustable. This is not arbitrary preference; it is the constitutional condition for the theory to be predictive of the world rather than fittable to it. We acknowledge that not every physicist will accept this elevation as constitutional — some will reasonably take the view that AX-4 is a strong methodological commitment that the cohort framework chooses to make and that other frameworks may decline. The cohort program stands by the commitment because the resulting predictions ($M_{\text{lock}} \approx 3.85$ TeV, $\Omega_m = 1/\pi$, kaxion at ~ 8.1 μeV , the integer OGN-5 ledger, the 28-to-zero parameter compression of the Standard Model) are sharp enough to be falsified within the 2026–2040 horizon. If the predictions fail, AX-4 will have been shown to be the wrong constitutional commitment, and the framework will have been refuted upstream.

N Convergences, empirical predictions, and structural questions**Objection 13 (O-BI6-13)**

The cohort framework claims that $U(1)_B$ is gauged in the bulk, protecting proton stability. This claim converges with a body of recent 4D phenomenology on gauged baryon number, notably the work of Davighi–Tong and related authors on flavor-universal anomaly-free gauged $U(1)_B$ extensions of the Standard Model. Does the cohort framework engage with this convergent independent program?

Response. we acknowledge the gap. The 4D phenomenology of gauged $U(1)_B$ — including the mass of the associated Z' boson, its LHC production cross sections, and the experimental constraints from LHC and IceCube — is the proper subject of GG-A7-Qi4 (Arisaka, 2026, A7-Qi4) and GG-A9-SM (Arisaka, 2026, A9-SM) rather than the present paper. The cohort framework’s constitutional derivation of gauged $U(1)_B$ in the bulk converges with the Davighi–Tong program at the 4D level: both arrive at gauged baryon number as a way to protect proton stability, but the cohort derivation is constitutional (forced by AX-2 (GGEP) of GG-A1-Constitution on the bulk side) while the Davighi–Tong derivation is phenomenological (chosen to fit 4D anomaly

cancellation and proton-stability constraints). We note the convergence, and the engagement has begun: the anomaly-free baryonic completion of Benakli, Goodsell, and Goudeau (Benakli, Goodsell, and Goudeau, 2026) — the concrete, benchmarked instance of this four-dimensional program — is compared with BI-6 structure by structure in Subsection 11.6, with the collider and dark-sector discriminants stated there. The remaining 4D phenomenology (production cross sections, LHC and IceCube constraints) stays the business of the boundary-spectrum papers of the cohort program.

Objection 14 (O-BI6-14)

The Executive Summary mentions $\alpha^{-1}(0) = 2^J + N^2 = 137$ with $(J, N) = (7, 3)$ of the OGN-5 ledger. Is this a derived identity or a numerological coincidence, given that the empirical value is $\alpha^{-1}(0) \approx 137.036$?

Response. the integer 137 is the leading-order constitutional prediction of the cohort fine-structure formula derived in detail in GG-A7-Qi4 (Arisaka, 2026, A7-Qi4) and GG-A9-SM (Arisaka, 2026, A9-SM). The 0.036 correction comes from higher-order contributions (vacuum-polarization thresholds, radiative corrections) that are themselves treated in the boundary-spectrum derivation of GG-A7-Qi4 and GG-A9-SM. The cohort framework predicts $\alpha^{-1}(0) = 137 + (\text{specific higher-order corrections})$; the integer 137 alone is not the full prediction. We do not claim integer 137 as a stand-alone identity; we claim the leading-order integer plus the derived corrections, with the full quantitative comparison to the empirical $\alpha^{-1}(0) \approx 137.036$ carried out in GG-A7-Qi4 / GG-A9-SM.

Objection 15 (O-BI6-15)

The kaxion frequency window is fenced by the same integer 35 that appears in the OGN-5 ledger as $K = M^2 + N^2 + L^2 = 35$. Is this an independent derivation or a fitted parameter?

Response. the integer $K = 35$ is derived in Section 9 from the Diophantine cascade of the OGN-5 ledger, and it appears *independently* in the scale-profile exponent of the GG-6 bulk geometry constructed in GG-A2-GG6 (Arisaka, 2026, A2-GG6); through that exponent it fences the kaxion’s frequency window in the companion kaxion paper. The two appearances of 35 are not coincidence: both descend from the same constitutional source (the OGN-5 ledger forced by BI-6 plus AX-4 (MSMF)). The window is therefore a derived fence, not a fitted parameter — though the absolute frequency inside it rests on a calibrated normalization (Open Problem O14). The empirical test is the detection of an $8.1 \mu\text{eV}$ ($\approx 1.95 \text{ GHz}$) signal — a single line — by the haloscope program assembling around the canonical window: IBS-CAPP scanning from below, and the planned ADMX-EFR band reaching its upper edge.

Objection 16 (O-BI6-16)

The cohort cosmological-constant prediction $m_\Lambda^2 = M_{\text{lock}}^2/M_{\text{Pl}} \approx 6 \text{ meV}$ is in the abstract. The observed cosmological constant gives $m_\Lambda \approx 2.3 \text{ meV}$, a factor of about 2.6–3 discrepancy. Doesn’t this falsify the prediction?

Response. the cohort framework predicts m_Λ at the leading-order seesaw factor. The factor-

of-3 discrepancy is acknowledged in GG-A10-Cosmos (Arisaka, 2026, A10-Cosmos) as Open Problem O6, and the expected resolution involves the closed-form derivation of the zero-mode overlap correction factor $\zeta_\Lambda \approx 1/50$ from the Wilson-line modulus and KK-tower calculation. The 6 meV figure should be read as “the cohort framework recovers 119 of the 120 orders of magnitude that the standard QFT vacuum-energy estimate fails by”; the remaining factor of approximately 3 is an explicit open quantitative problem rather than a constitutional success. We acknowledge that until the $\zeta_\Lambda \approx 1/50$ derivation is complete, the cosmological-constant claim should be read as “the cohort framework dramatically improves on the QFT vacuum-energy estimate but does not yet hit the observed value to within experimental precision”.

Objection 17 (O-BI6-17)

The cohort framework treats $\text{Spin}(1,5)$ as a “gauge factor” in the composite group $G_{\text{Bulk}} = \text{Spin}(1,5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Qi}$. In standard 6D supergravity treatments, $\text{Spin}(1,5)$ is the local Lorentz group of the frame bundle, distinct from the matter-gauge factors. Isn’t the cohort’s lumping-together of these structurally distinct objects technically non-standard?

Response. yes, the composite-connection treatment of AX-2 (GGEP) is non-standard at the notational level. Under AX-2, gravity and gauge are not separate fundamental sectors but algebraic facets of one composite connection on the 6D manifold $\mathcal{M}_6 = \mathcal{M}_4 \times I_r \times S_\theta^1$. The spin connection (the gravity component) and the gauge connections (for $\text{SU}(5)$, $\text{U}(1)_B$, $\text{U}(1)_{Qi}$) are projections of the same composite primitive. The standard treatment, which separates the spin connection from the gauge connections, corresponds in the cohort framework to the GGEP-closed limit in which the composite structure is not yet recognized. The $\text{Spin}(1,5)$ factor in G_{Bulk} is the gravity component of the composite connection, not an additional gauge factor in the usual particle-physics sense. This is constitutionally coherent with AX-2 but admittedly unfamiliar to readers trained on the standard separated-sector treatment of 6D supergravity. The reconciliation with the standard treatment is the natural subject of follow-up exposition; the present paper takes the composite-connection treatment as a constitutional input from GG-A1-Constitution (Arisaka, 2026, A1-Constitution) and proceeds from there.

O The local descent and the four-form X_4 : proof internals

Cluster: the engine of Part II, audited form by form. The objections so far probe the framework’s posture; the next seven probe the descent computation that does the anomaly-canceling work. A reader who accepts the constitutional method will still demand that each step of the BI-6 descent be gauge-invariant, sign-controlled, and integral.

Objection 18 (O-BI6-18) (the differential geometer)

The descent chain $I_8 \rightarrow I_7^{(1)} \rightarrow I_6$ is built from Chern–Simons forms, which are not globally defined. So the “local descent” proof of Theorem 5.5 is gauge-dependent at

the level of forms. How is the cancellation gauge-invariant rather than an artifact of a local trivialization?

Response. The objection correctly identifies that the descent forms $I_7^{(1)}$ and the Chern–Simons potentials of Eq. (37) are defined only patch-by-patch, and that no statement about them is physical until it is promoted to a globally well-defined object. That promotion is exactly what the gerbe / differential-cohomology lift of Section 5.2 supplies: the modified Bianchi identity $dH_3 = X_4$ is read as the curvature equation of a connection on an abelian gerbe, and the Green–Schwarz term becomes a globally defined holonomy rather than a local integral of B_2 . The local descent of Theorem 5.5 establishes the *perturbative* (infinitesimal-gauge) cancellation; its gauge-invariant content is precisely the statement that the descent forms glue into the gerbe class of Section 5.2, and the global completion (Theorem 6.2) is the statement that this class is trivial. So the local descent is not claimed to be gauge-invariant by itself; it is the first half of a two-step argument whose second half is the gerbe-level globalization. A reader who stops at the local descent is right to call it trivialization-dependent — and is pointed to the very next section for the cure.

Objection 19 (O-BI6-19) (the anomaly specialist)

In $X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - \frac{2}{5}F_B^2 - \frac{2}{5}F_{Qi}^2$, the relative *signs* between the gravitational and the gauge pieces are what make the factorization cancel. Are those signs derived, or chosen so that the cancellation works?

Response. The signs are not free. The framework’s stated position (the geometric-origin analysis of Section 2) is that the *magnitudes* of the four coefficients are fixed geometrically — by the trace normalizations of the $\text{Spin}(1, 5)$, $\text{SU}(5)$, $\text{U}(1)_B$, $\text{U}(1)_{Qi}$ blocks — while the *signs* are fixed by anomaly cancellation itself, i.e. by the requirement that the one-loop hexagon anomaly I_8 factorize as a perfect square (Eq. (30), Eq. (6)). This is the honest division of labor: geometry sets $|\text{coeff}|$, the cancellation condition sets $\text{sign}(\text{coeff})$. The objection’s worry — that signs were reverse-engineered — is therefore half right and half answered: the signs *are* the ones that make cancellation work, but that is not circular, because the same sign assignment is forced by the independent requirement that the gravitational $\text{tr } R^2$ piece have the orientation dictated by the chirality of the boundary fermions. A sign flip on any gauge block would violate the chirality-fixed gravitational anomaly inflow of Section 2, not merely spoil a convenient cancellation.

Objection 20 (O-BI6-20) (the topologist)

You factor the hexagon anomaly as $I_8 = \frac{1}{2}X_4 \wedge X_4$ — a perfect square. But a generic degree-8 invariant polynomial does *not* factor as the square of a 4-form. Isn’t the factorizability itself the nontrivial miracle, and aren’t you presenting it as if it were automatic?

Response. Factorizability is indeed the nontrivial content, and the paper’s claim is precisely that the constitution *forces* it rather than that it is automatic. Theorem 5.1 is the statement that does this work: the microscopically computed 4×4 BI-6 anomaly matrix (Eq. (43)) collapses to *rank*

one, and a rank-one symmetric form is exactly a perfect square $v \otimes v$, with $v = (1, -1, -\frac{2}{5}, -\frac{2}{5})^T$ the unique (up to sign) eigenvector. So the factorization $I_8 = \frac{1}{2}X_4 \wedge X_4$ is not assumed; it is the corollary of the computed rank (Proposition 7.5, Eq. (20)). The miracle the objection points to is real, and the paper’s answer is that the miracle is *measured*: the rank of the anomaly matrix is computed from the SU(5) representation data of Table 18 and found to be one. Had it come out rank two, the theory would have needed a second tensor (Objection 5) and would have been inadmissible under AX-4. The single-tensor Green–Schwarz factorization is thus a constitutional prediction that the rank-one computation confirms.

Objection 21 (O-BI6-21) (the Chern–Weil specialist)

The gerbe lift with $dH_3 = X_4$ requires X_4 to have *integral* periods — a flux-quantization condition. But your coefficients include the fraction $\frac{2}{5}$. How can a 4-form with fractional coefficients represent an integral cohomology class?

Response. This is the sharpest mathematical objection in the cluster, and the answer is the quadratic-integrality condition of Eq. (275). The fractional $\frac{2}{5}$ is a coefficient in the *normalized-trace* basis; the physically relevant class is the one paired against the integral homology of the bulk, and the $U(1)_B$, $U(1)_{Q_i}$ field strengths carry the compensating integer charges (the OGN-5 ledger) so that the product X_4 lands on an integral class. Concretely, Section 5.2 shows that the characteristic-class combination of Eq. (36) is integral precisely because the $\frac{2}{5}$ is multiplied by the squared $U(1)$ charges, which are integers divisible by 5 on the relevant lattice. The fraction is an artifact of writing the gravitational and SU(5) pieces in a unit-normalized basis; in the charge-integral basis the periods are integers. The objection is exactly the check that has to be done, and Eq. (275) is where it is done; if it failed, the gerbe lift would not exist and the global theorem would be vacuous.

Objection 22 (O-BI6-22) (the skeptic of “ten coefficients”)

You make much of a “ten-coefficient” anomaly table. But a symmetric 4×4 matrix has exactly 10 independent entries — of course there are ten. Isn’t “ten coefficients” just the generic count of a symmetric matrix, dressed up as structure?

Response. The objection is right that $10 = \binom{4+1}{2}$ is the generic entry-count of a symmetric 4×4 form, and the paper does not claim the *number* ten is surprising. What is structural is not that there are ten entries but that those ten entries (Table 14, Eq. (43)), read off the coefficient vector, are mutually consistent with *rank one* — i.e. all ten are determined by the four components of a single eigenvector and must satisfy six independent rank-one constraints among themselves. The content is the six vanishing 2×2 minors, not the ten entries. A generic symmetric form with ten free entries would have rank four and would need four tensors; the computed form has rank one. So “ten coefficients” is the bookkeeping, and “rank one among the ten” (Theorem 5.1) is the physics. We take the point: the load-bearing fact is the rank, not the entry-count, and the presentation foregrounds it accordingly.

Objection 23 (O-BI6-23) (the representation theorist)

The trace identities $\text{Tr } F_5^2$, $\text{tr } R^2$, and the rest depend on the fermion representation con-

tent. Your anomaly polynomial appears to use one generation. Three generations triple the gauge anomaly but not the pure-gravitational $\text{tr } R^4$ piece. Does the factorization survive the actual generation count?

Response. It survives, and the mechanism is the gravitational-closure constraint of Section 7.5 (Appendix H). The reducible (factorizable) part of the anomaly — the gauge and mixed gauge-gravitational pieces of Eq. (33) — scales with the generation number, but it scales *uniformly* across the $X_4 \wedge X_4$ factorization because all three generations carry the same SU(5) charges, so the trace-ratio $2/5$ (Eq. (31)) is generation-independent and the rank-one structure is preserved. The *irreducible* $\text{tr } R^4$ piece, which does not factorize and cannot be canceled by Green–Schwarz, is handled separately by the anti-self-dual-tensor / singlet-fermion linear constraint of Appendix H; that constraint is where the generation count and the singlet sector enter, and its solution ($n_{SD} = 100$, $n_F = 800$ on the $\Omega = 25$ branch) is fixed once the visible spectrum — three generations of SU(5) matter — is specified. So the generation count is not ignored; it is the input to the gravitational-closure system, and the factorization is generation-robust because the trace-ratio is.

Objection 24 (O-BI6-24) (the boundary-term auditor)

The Green–Schwarz variation $\delta S_{GS} \sim \int B_2 \wedge \delta(\cdots)$ is integrated over the bulk $\mathcal{M}_4 \times I_r \times S_\theta^1$, and I_r has two endpoints. Integration by parts in the descent produces boundary terms at those endpoints. Do they vanish, or is the cancellation only valid on a closed manifold?

Response. The boundary terms are real and are treated, not assumed away. The Green–Schwarz variation of Eq. (155) is integrated over the product geometry, and the integration-by-parts in the descent does generate endpoint contributions at the ultraviolet and infrared ends of I_r . Two facts dispatch them. First, the compact factor S_θ^1 contributes no boundary (it is closed), so only the radial ends matter. Second, the intended-class definition of Section 6 imposes the boundary conditions under which the radial endpoint variations of B_2 vanish — the same conditions that make the one-way renormalization-group flow of POS-5 well-defined — so the surviving anomaly is the bulk descent term alone. We concede that the present paper states these as part of the intended-class boundary data rather than deriving their uniqueness; the uniqueness of the boundary conditions is part of the open agenda of Open Problem O10 (Section 17). What is established is that, with the stated boundary data, the endpoint terms cancel and the bulk factorization is the whole story.

P Global completion: Dai–Freed, cobordism, the eta-invariant

Cluster: how complete is “global anomaly freedom”? The most important objections to a 6D consistency claim are about the global, non-perturbative anomaly. We let the strongest versions stand and mark exactly where the proof is structural and where it is open.

Objection 25 (O-BI6-25) (the Dai–Freed purist)

Modern global-anomaly freedom means the partition function is a genuine *function*, not a section of a line bundle — equivalently, the Dai–Freed eta-invariant must be trivial on every bounding seven-manifold. You verify “four hypotheses (H1)–(H4).” Are those equivalent to Dai–Freed triviality, or strictly weaker?

Response. They are the hypotheses under which the Dai–Freed / Hopkins–Singer anomaly field theory is trivial, but the paper is careful (Theorem 6.2) to claim only what the hypothesis-check delivers. (H1)–(H4) are the bordism-theoretic conditions (Dai and Freed, 1994, Hopkins and Singer, 2005, Monnier and Moore, 2019) whose satisfaction implies that the exponentiated eta-invariant of Eq. (277) is trivial on the intended background class; verifying them is logically equivalent to Dai–Freed triviality *restricted to that class*. What the paper does *not* do — and says so — is evaluate the eta-invariant explicitly on each mapping torus, the computation Monnier–Moore carry out for $\mathcal{N} = (1, 0)$ supergravity; that computation is now fixed by theorem rather than left open: Subsection 6.4 proves the refinement unique and computes the phase exactly (Open Problem O9, Resolved). So the honest scorecard: triviality is established on the intended class via the hypothesis route, and the per-torus eta-invariant evaluation is an error-catching cross-check, no longer a confirmation the claim waits on. The objection’s distinction between “function” and “section” is exactly the right one, and the paper’s claim sits on the function side for the intended class, contingent on (H1)–(H4) holding — which it checks.

Objection 26 (O-BI6-26) (the cobordism theorist)

A global anomaly can hide in a torsion class of the bordism group $\Omega_7^{\text{Spin}}(BG)$ for $G = \text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$. Have you actually computed that group, or assumed it vanishes?

Response. We have not computed $\Omega_7^{\text{Spin}}(BG)$ in closed form, and the paper does not claim to. What Theorem 6.2 establishes is that the anomaly theory is trivial on the intended background class via the (H1)–(H4) check; a full computation of the bordism group — which would establish triviality on *all* backgrounds with that gauge structure, not merely the intended class — is precisely the content of Open Problem O10 (Section 17). The objection identifies the right invariant: a nonzero torsion class in $\Omega_7^{\text{Spin}}(BG)$ paired nontrivially with the anomaly theory would host a global anomaly. The paper’s position is that on the physically realized class (product geometry, pullback bundle, single tensor) the relevant pairing is checked to vanish, and that the exhaustive bordism computation is a well-posed follow-up that the modern cobordism technology (Brennan, Carta, and Vafa, 2017, Kaidi, Tachikawa, and Yonekura, 2025) could carry out. We flag this as a genuine limitation, not a closed result.

Objection 27 (O-BI6-27) (the eta-invariant skeptic)

Without supersymmetry there is no multiplet structure to organize the spectrum, and the index-theorem shortcuts that make Monnier–Moore tractable rely on the $\mathcal{N} = (1, 0)$ field content. For the non-supersymmetric cohort spectrum, how is the eta-invariant even computable in principle?

Response. The eta-invariant is a property of a Dirac operator coupled to a background, and is

defined whether or not the spectrum is supersymmetric — SUSY makes the *evaluation* tractable, not the *definition* possible. The bordism-theoretic hypotheses (H1)–(H4) of Theorem 6.2 are, as Section 6.7 stresses, topological and do not invoke supersymmetry; they go through for the cohort’s chiral content. What SUSY buys in Monnier–Moore is the explicit closed-form partition function on bounding manifolds, expressed through the SUGRA multiplet data; the non-SUSY cohort spectrum (three SU(5) generations plus the $n_F = 800$ opposite-chirality singlets of the $\Omega = 25$ branch) requires a direct eta-invariant computation with no multiplet shortcut. That computation was why Open Problem O9 was hard; Subsection 6.4 now discharges it structurally — an APS index per species and a vanishing filling signature (Lemma I.10). So the answer is: computable in principle, computed structurally in fact, with the per-torus numerical re-run remaining as an error-catching cross-check (Open Problem O9, Resolved).

Objection 28 (O-BI6-28) (the global-gauge-anomaly checker)

Witten-type global gauge anomalies live in homotopy groups such as $\pi_6(\text{SU}(5))$. A perturbatively anomaly-free theory can still have a nonzero π_6 global anomaly. Have you checked that the 6D gauge group hosts no such Witten anomaly?

Response. The relevant homotopy obstruction is part of the (H1)–(H4) verification rather than a separate omitted check, and it deserves to be explicit. In six dimensions the Witten-type global gauge anomaly is governed by π_6 of the gauge group and, more completely, by the torsion pairing of the cobordism description (Objection 26). For the cohort gauge group the homotopy check is clean: Bott periodicity gives $\pi_6(\text{SU}(n)) = 0$ for $n \geq 4$, hence $\pi_6(\text{SU}(5)) = 0$, and $\pi_6(\text{U}(1)) = 0$, so no independent Witten-type obstruction exists (the Lemma for H2 of Appendix I; contrast SU(2) and SU(3), where π_6 is $\mathbb{Z}_{12}$ and $\mathbb{Z}_6$ respectively and the check would be non-trivial). The fully rigorous statement is the cobordism one, which is Open Problem O10. So: the perturbative anomaly is canceled by the rank-one factorization (Theorem 5.1), the π_6 -type obstruction vanishes identically for the cohort gauge group, and the exhaustive homotopy/cobordism sweep is the open completion. The objection names a check that must not be skipped, and the paper’s claim is that it passes on the intended class and is open in full generality.

Objection 29 (O-BI6-29) (the topology-sum critic)

Restricting to the “intended background class” means your global theorem says nothing about topology change, non-trivial S^1_θ bundles, or baby universes. A complete quantum-gravity consistency proof must sum over all topologies. Isn’t the restricted theorem therefore not a quantum-gravity consistency proof at all?

Response. The restricted theorem is a consistency proof for the *effective field theory on the intended background*, not a proof of full non-perturbative quantum gravity, and the paper should not be read as claiming the latter. GG-Theory is, by its own account (Section 4.1), an effective description at scales below the six-dimensional Planck scale; the anomaly-freedom theorem certifies that this EFT is internally consistent on the backgrounds it actually uses. Summing over all topologies — the gravitational path integral with baby universes and topology change — is a question about the UV completion that the cohort does not claim to possess (it is the UV-completion open problem inherited from (Arisaka, 2026, A2-GG6)). The statement is

therefore: BI-6 proves anomaly freedom of the GG-Theory EFT on its physical background class; it does not prove, and does not claim, consistency of a full topology-summed quantum gravity. A reader who demands the latter is asking a question the whole cohort marks as open, and is pointed to Open Problem O10 and the UV-completion discussion of Section 17.

Objection 30 (O-BI6-30) (the honest-status auditor)

Your global theorem is “structural” (hypotheses checked) rather than “explicit” (partition functions evaluated on each mapping torus). A skeptic concludes: you have not *proved* global anomaly freedom; you have *reduced* it to hypotheses you have not numerically verified. Is that fair?

Response. It is fair as a description and unfair as a verdict, and the distinction matters. Fair: Theorem 6.2 is a conditional-then-discharged statement of the form “(H1)–(H4) $\Rightarrow$ global triviality, and (H1)–(H4) hold on the intended class,” where the discharge of (H1)–(H4) is by structural/topological argument rather than by explicit mapping-torus evaluation. So yes, the proof is a reduction-plus-structural-discharge, not an explicit numerical computation. Unfair: a reduction to verified topological hypotheses *is* a proof in the differential-cohomology framework — it is exactly the form Monnier–Moore’s own theorem takes before its SUGRA-specific numerical illustration. The per-torus eta-invariant evaluation (O9, since Resolved by Subsection 6.4) is an independent error-catching re-run, valuable but not load-bearing for the structural claim. The paper’s standing is therefore: global anomaly freedom is proved on the intended class modulo the topological hypotheses, which are checked — with the defining refinement proved unique and the phase computed exactly. We prefer to state this precisely and absorb the skeptic’s discount than to overclaim.

Q The OGN-5 ledger: theorem, or target?

Cluster: is the integer ledger derived or reverse-engineered? Part III calls the five integers “a theorem, not an input.” The next six objections press that claim from the number-theoretic side.

Objection 31 (O-BI6-31) (the number theorist)

The dual lock $K = LJ = M^2 + N^2 + L^2 = 35$ with $(M, N, L, J, K) = (1, 3, 5, 7, 35)$ is called the unique *small*-integer solution. “Small” is doing the work. What is the precise bound, and why is it constitutional rather than chosen to exclude the larger solutions that surely exist?

Response. The precise statement is in Theorem 9.3 and Appendix G, and “small” is not a vague aesthetic but the output of AX-4 (MSMF): among integer solutions of the joint system Eqs. (78)–(81), MSMF selects the minimum-rank one, which fixes $M = 1$ (the primitive tensor slot), $L = 5$ (smallest simple group containing the Standard Model), and $N = 3$ (the trace-ratio integer). Given $(M, N, L) = (1, 3, 5)$, the remaining pair (J, K) is forced uniquely by $K = LJ = M^2 + N^2 + L^2$ to be $(7, 35)$ — there is no freedom left, large or small. So the honest

logic is not “pick the smallest of many solutions” but “MSMF fixes three integers, and the two closure equations then fix the other two uniquely.” The larger solutions the objection has in mind $((M, N, L) = (2, 6, 10), \dots)$ are exactly the unforced structural enlargement that MSMF forbids (Objection 8). The bound is therefore not an arithmetic cutoff but a constitutional minimality clause, and Appendix G states it as such.

Objection 32 (O-BI6-32) (the “two equations, five unknowns” critic)

Two Diophantine equations cannot fix five integers. You supply $M = 1$, $L = 5$, $N = 3$ from outside the cascade (dimensional minimality, $SU(5)$, the trace ratio). So the “five-integer derivation” is really three external inputs plus two consistency checks. Concede?

Response. The cascade is sequential, not simultaneous, and that much is granted. But the objection’s arithmetic no longer holds. $M = 1$ comes from dimensional minimality (Arisaka, 2026, A2-GG6); $L = 5$ from single-tensor Green–Schwarz consistency (Theorem 9.2), which is free of N at every stage and no longer appeals to the smallest-simple-group clause; $N = N_c = 3$ from the closure quadratic $N^2 - 10N + 21 = 0$ together with the block embedding $N < L$, which excludes the second root $N = 7$ (Theorem 9.4) — the mirror of the $L = 2$ exclusion in Appendix E; and $J = 2N + 1$ is the closure-index clause. So *no* integer of the ledger is an empirical input, and the trace ratio, which would have consumed the 3+2 split, is used only as a check. One thing we must give up in exchange, and say so plainly: because the dual lock now carries N , its two routes are required to agree rather than observed to agree, and we no longer claim the agreement as a free confirmation. What the constraint buys instead is content — it forces the ledger onto $N = L \pm \sqrt{L-1}$, so that $L-1$ must be a perfect square and $L \in \{2, 5, 10, 17, 26, \dots\}$ — and $L = 5$ then arrives independently. The accurate description is therefore: two prior theorems, one closure determination, one structural clause, and a constitutional minimality axiom AX-4 that carries the whole weight and is named.

Objection 33 (O-BI6-33) (the BI-6 $\leftrightarrow$ OGN-5 bridge critic)

You claim BI-6 forces the OGN-5 ledger. But BI-6 is a statement about a 4-form and OGN-5 is an integer tuple; the only bridge is the trace ratio $2/5$. Is that bridge a theorem, or an identification you impose to connect the two halves of the paper?

Response. It is Theorem 8.1, and the bridge is the rank-one eigenvector, not a free identification. The BI-6 factorization forces the coefficient vector $v = (1, -1, -\frac{2}{5}, -\frac{2}{5})$ (Theorem 5.1); the ratio of its gauge entry to its gravitational entry is $\frac{2}{5} = (L - N)/L$, which is an equation relating the very integers L and N that the OGN-5 ledger contains. So the trace ratio is a computed group-theoretic quantity read as a Diophantine constraint on the OGN side. But we should be exact about what computes it, because the objection is closer to the mark than the previous sentence allowed. The rank-one condition fixes the *direction* of v and, with the primitive normalization, its gravitational and non-Abelian entries; it does not fix the Abelian entry, and the microscopic matrix carries that entry because the benchmark spectrum was constructed against a target already containing it. What computes the Abelian entry is Theorem 8.1, the trace ratio $\text{Tr}(B_0^2)/\text{Tr}(Y_0^2)$ evaluated on the block split of the defining representation — and

that evaluation presupposes the split it is later used to determine. Eq. (31) is where the same number appears in both languages. The objection is right that this single ratio is the entire load-bearing link between Parts II and III; the paper’s claim is that the link is a computed equality, not a postulated correspondence, and that is exactly why Theorem 8.1 is stated as the hinge of the paper.

Objection 34 (O-BI6-34) (the $\Omega = 25$ brancher)

The gravitational closure selects the “ $\Omega = 25$ minimal branch” with $n_{SD} = 100$, $n_F = 800$. Why $\Omega = 25$? Is there an $\Omega = 24$ or $\Omega = 26$ branch, and what constitutional clause picks 25 rather than its neighbors?

Response. Ω is the integer that labels the discrete solution branches of the gravitational-closure linear system (Eq. (247), Appendix H), and neighboring integer values $\Omega = 24, 26$ would solve the gravitational 2×2 system alone ($n_{SD} = 4\Omega$ is an integer for every integer Ω); it is the gauge-sector integrality of Theorem 7.8 that restricts to $\Omega \in 25\mathbb{Z}$. What selects $\Omega = 25$ is the same AX-4 (MSMF) minimality plus consistency with the OGN-5 master integer $K = 35$: the physical benchmark branch of Eq. (256) is the minimal-singlet solution compatible with the scale-depth integer, and $\Omega = 25 = L^2$ is the value that ties the gravitational sector to the $L = 5$ gauge integer rather than floating free. So $\Omega = 25$ is not chosen by hand; it is the branch that the minimality clause and the $K = 35$ lock jointly pick out, with the neighbors 24, 26 excluded as either non-minimal or inconsistent with the ledger. We concede the branch structure is one of the more intricate parts of the gravitational closure; the $\Omega = 25 = L^2$ tie is made explicit in Appendix H.

Objection 35 (O-BI6-35) (the 800-singlet skeptic)

The gravitational closure needs $n_F = 800$ gauge-singlet Weyl fermions. That is a large, specific, and *unobserved* sea of particles. Where do they live in the boundary spectrum, and is an invisible 800-fold singlet sector not an embarrassment?

Response. The singlets are gauge-singlet and bulk-localized, and their boundary fate is a question the present paper hands to the spectrum papers rather than answers. Appendix H fixes $n_F = 32\Omega = 800$ as the count required to cancel the irreducible $\text{tr } R^4$ anomaly on the $\Omega = 25$ branch (exactly eight singlets per anti-self-dual tensor); being gauge singlets, they carry no Standard-Model charge and so are not excluded by collider or precision-electroweak data. Their physical disposition — whether they are heavy bulk modes lifted near the six-dimensional Planck scale, sterile states relevant to the boundary neutrino sector, or part of the dark sector — is determined by the boundary-spectrum derivation of GG-A9-SM (Arisaka, 2026, A9-SM) and GG-A5-GDOS (Arisaka, 2026, A5-GDOS), not here. We acknowledge the objection’s discomfort honestly: an 800-fold singlet sector is a large prediction, and if the spectrum papers cannot find a phenomenologically viable home for it, that is a genuine vulnerability of the construction. The present paper’s claim is narrower — that anomaly cancellation *requires* exactly this count — and the count is therefore a falsifiable structural prediction, not a free parameter.

Objection 36 (O-BI6-36) (the “theorem, not an input” skeptic)

The Part III subtitle calls the integer ledger “a theorem, not an input.” That is rhetorically loaded: a theorem needs axioms, and your axiom AX-4 is exactly the one tuned so that the theorem yields these integers. Isn’t “theorem, not input” just circular packaging?

Response. The slogan means something precise and defensible, though the objection is right that it can be over-read. “Theorem, not an input” is the claim that the integers (1, 3, 5, 7, 35) are *not* adjustable dials of the model — you cannot change L from 5 to 6 and refit, because L is fixed by the smallest-simple-group clause and the trace ratio, and every other integer follows (Section 8). Contrast a conventional model where Yukawa couplings or compactification data are inputs chosen to fit. The honest part the objection captures: this is a theorem *relative to AX-4*, and AX-4 is a constitutional choice, not a logical necessity (Objection 12). So the non-circular content is “given the constitution, the integers are forced”; the circular reading the objection warns against would be “the integers prove the constitution,” which the paper does not claim. We take the wording seriously: the subtitle asserts the integers are outputs of the axioms, and the place to contest it is AX-4 itself, not the arithmetic that follows from it.

R String theory: what exactly is the relationship?

Cluster: Part IV audited by its target. The comparison Part concedes “shared mathematics, structurally different physics.” The next seven objections are the strongest a string theorist will raise.

Objection 37 (O-BI6-37) (the string theorist)

Your $X_4 = \text{tr } R^2 - \text{Tr } F^2$ is literally the heterotic Green–Schwarz 4-form dimensionally reduced to six dimensions. So BI-6 is heterotic Green–Schwarz on a 6D slice. What is new beyond bookkeeping?

Response. The mathematical form is shared and the paper says so plainly (Section 10, Eqs. (27) and (28)); what is new is not the Green–Schwarz move but its *origin* and its *rigidity*. In the heterotic string, X_4 descends from worldsheet modular invariance and the gauge group is forced to $E_8 \times E_8$ or $SO(32)$ by that same string consistency. In GG-Theory there is no string and no worldsheet; X_4 is forced by the constitutional clauses (AX-2 read as the modified Bianchi identity, Eq. (16)) and the gauge group is forced to the *smallest simple group containing the Standard Model*, $SU(5)$, by AX-4. The new content is that a four-component X_4 with the specific trace-ratio lock $2/5$ and the OGN-5 integer ledger is forced *without* string input, and that this rigidity yields integer predictions ($K = 35$, $\alpha^{-1} = 137$ at leading order) that the heterotic construction, with its moduli, does not. So the bookkeeping is shared; the claim that the bookkeeping is *constitutionally forced and moduli-free* is the difference, and it is a physical difference because it is what makes the integers predictions rather than choices.

Objection 38 (O-BI6-38) (the heterotic specialist)

String consistency picks $E_8 \times E_8$ and $SO(32)$ for a reason: lower-rank groups like $SU(5)$ famously can carry a global Witten anomaly that the string groups avoid. By choosing $SU(5)$ on constitutional grounds, haven't you walked straight into the anomaly the string groups were selected to dodge?

Response. The comparison table of forbidden alternatives (Table 15) addresses exactly this, and the answer turns on the difference between ten and six dimensions. The string groups $E_8 \times E_8$, $SO(32)$ are selected in *ten* dimensions by the $\dim G = 496$ Green–Schwarz condition together with modular invariance; that condition is a 10D statement and does not apply in six dimensions, where the anomaly polynomial and the factorization condition are different (Eq. (28)). In 6D the relevant global obstruction for $SU(5)$ is the π_6 -type anomaly of Objection 28, and the cohort's chiral content is arranged — by the rank-one structure (Theorem 5.1) — so that it cancels on the intended class, with the exhaustive cobordism check flagged as Open Problem O10. So the framework has not ignored the constraint that picks the string groups; it has observed that that constraint is dimension-specific, and has discharged the 6D analogue by the rank-one factorization rather than by enlarging the group. Whether the cobordism sweep confirms this in full generality is the open completion, and it is named.

Objection 39 (O-BI6-39) (the F-theory geometer)

F-theory realizes $SU(5)$ grand unification geometrically, with all anomalies automatically canceled by the geometry of the elliptic fibration, and it is UV-complete. Why is the cohort's constitutional $SU(5)$ preferable to F-theory's geometric $SU(5)$, which already does the anomaly job and embeds in string theory?

Response. It is not “preferable” in UV-completeness — F-theory wins there, and the paper does not contest it (Section 11.2, Table 7). The difference the cohort claims is parametric rigidity. F-theory's $SU(5)$ comes with the moduli of the elliptic fibration — complex-structure and Kähler parameters, flux choices (Morrison and Taylor, 2012a, Morrison and Vafa, 1996a) — which are exactly the continuous freedom that places it in the landscape and prevents it from predicting, say, α^{-1} or the scale-depth integer. The cohort's $SU(5)$ is fixed with *no* fibration moduli, because AX-4 forbids them; the price is the absence of a UV completion (conceded, inherited from (Arisaka, 2026, A2-GG6)), and the payoff is integer predictions. So the honest trade is: F-theory offers UV-completeness plus moduli; the cohort offers moduli-freedom plus integer predictions but no UV completion. They are not competing on the same axis. A reader who weights UV-completeness above predictivity should prefer F-theory; the cohort is a bet that moduli-free predictivity is the more falsifiable virtue.

Objection 40 (O-BI6-40) (the holographer)

Your warped throat is Randall–Sundrum, and RS has a known AdS/CFT dual. Does BI-6 admit a holographic dual description? If not, why is the warp physical rather than a redundant dual artifact?

Response. The AdS/RS comparison (Section 12, Table 11, Eq. (115)) is drawn precisely so this

question can be answered cleanly. The warp factor of the GG-6 bulk is geometrically Randall–Sundrum-like, and to the extent that the RS throat has an AdS/CFT reading (Maldacena, 1998, Randall and Sundrum, 1999), the cohort inherits a holographic *intuition* for the radial direction as a renormalization-group coordinate. But BI-6 is not a statement that requires a dual to be meaningful: it is the modified Bianchi identity of a bulk gerbe (Section 5.2), a bulk-geometric fact about anomaly inflow, and the anomaly it cancels is a property of the boundary chiral fermions registered at the infrared end. The warp is physical because it carries the one-way RG semigroup of POS-5, an irreversible registration arrow that a time-symmetric AdS/CFT duality does not by itself supply (this is the OGGEF-vs-AdS/CFT distinction developed in the cohort’s bulk paper). So: a holographic dual description is compatible and partially illuminating, but BI-6 stands as a bulk fact independent of whether a dual is exhibited, and the warp’s physical content is the registration arrow, not a dual artifact.

Objection 41 (O-BI6-41) (the swampland theorist)

The cobordism conjecture does more than demand a trivial cobordism class — it predicts specific defects and branes that must exist to trivialize otherwise-nontrivial classes. Does the cohort predict those cobordism defects, or does it only assert that the class is trivial?

Response. At present the paper asserts triviality on the intended class (Section 13, Section 6) and does *not* enumerate the cobordism defects the conjecture would associate with any nontrivial classes — that enumeration is part of the open cobordism agenda (Open Problem O10). The objection identifies a genuine and currently-unmet obligation: the modern form of the cobordism conjecture (Kaidi, Tachikawa, and Yonekura, 2025, McNamara and Vafa, 2019) ties anomaly-freedom to a defect spectrum, and a complete treatment would either exhibit the required defects or show the relevant classes vanish so that no defects are needed. The cohort’s position is the second (the intended class is arranged to have trivial cobordism pairing), but the exhaustive check — and hence the question of whether any defects are predicted — is not done. We flag this as one of the sharper swampland-side open problems and do not claim the defect spectrum as a result.

Objection 42 (O-BI6-42) (the landscape pragmatist)

You say AX-4 collapses the 10^{500} landscape to a single vacuum. But the landscape is a feature of *flux compactification*; a 6D theory with one tensor and no fluxes has simply not turned on the moduli that generate the landscape. Is the “collapse” a real result or an artifact of not including the structures that produce the landscape in the first place?

Response. The objection is sharp and the answer must concede part of it. The cohort does not derive a single vacuum from *within* a flux-compactification framework by stabilizing all moduli; rather, AX-4 forbids the unforced continuous structures — extra tensors, flux integers beyond the OGN-5 ledger, fibration moduli — whose proliferation *is* the landscape (Table 7, Section 10). So in one sense the objection is right: the cohort avoids the landscape by not admitting the structures that generate it, not by stabilizing them. Where we push back: that is

precisely the claim — that those structures are unforced and therefore inadmissible under the constitution, not that they are present-but-stabilized. The framework is making a bet that the landscape is an artifact of admitting unforced freedom, and that a constitution forbidding such freedom is more predictive. Whether that bet is right is empirical: if the integer predictions hold, the moduli-free posture is vindicated; if a continuous modulus is needed to fit data, AX-4 fails (Objection 12). The “collapse” is thus a constitutional stance with falsifiable consequences, not a theorem that the flux landscape is empty.

Objection 43 (O-BI6-43) (the “shared mathematics” auditor)

Part IV’s own subtitle concedes “shared mathematics, structurally different physics.” But if the mathematics is shared, the anomaly-freedom *content* is identical, and the “structural difference” is interpretation, not physics. Rebut.

Response. The rebuttal is that “the mathematics is shared” is true at the level of the Green–Schwarz *move* and false at the level of what is forced. The shared mathematics is the factorization mechanism: given an X_4 , the descent and the tensor coupling cancel the reducible anomaly identically in string theory and in GG-Theory (Section 11, Section 10.7). The non-shared, non-interpretive physics is *which X_4 is allowed*: string theory permits a family of X_4 ’s parametrized by the compactification, while GG-Theory forces the unique four-component X_4 with the 2/5 lock and the integer ledger. That is a difference in the space of admissible theories, not a difference in interpretation of a single theory — it changes which numbers are predicted. So the subtitle is accurate: the cancellation calculus is shared (mathematics), but the constraint that selects one X_4 out of a family is the constitution (physics), and it has different empirical consequences (integer predictions vs. a landscape). The objection would be right if the cohort merely re-interpreted a string vacuum; it does not — it forbids the family the string admits.

S Constitutional framing, method, and self-sufficiency

Cluster: what does an anomaly proof “constitutionalized” actually add? The next five objections press on the framing and on what A3 establishes on its own.

Objection 44 (O-BI6-44) (the philosopher of physics)

Anomaly freedom is a consistency requirement of *any* quantum gauge theory. By calling it “forced by the constitution,” you have relabeled a necessary condition as a constitutional achievement. What is the added content beyond “the theory is consistent”?

Response. The added content is that the constitution forces a theory that *is* anomaly-free, rather than merely requiring that whatever theory we write down be checked for anomalies. The distinction is between a filter and a generator. In the conventional approach, anomaly freedom is a constraint one imposes after choosing a model, discarding the anomalous candidates; the space of inputs (gauge group, matter content) is chosen by hand and then filtered. In GG-Theory, the

gauge group and matter content are *outputs* of AX-1–AX-4 (the bulk theorems of GG-A2-GG6 (Arisaka, 2026, A2-GG6), together with Theorem 6.22 and the OGN-5 theorem of Section 9), and the theorem of Part II is that these forced outputs are automatically anomaly-free with a single tensor and the integer ledger. So the claim is not “our theory passes the anomaly check” (true of any consistent theory) but “the constitution that fixes the theory also guarantees it passes, and fixes the cancellation structure uniquely.” The content is the uniqueness and the no-tuning, not the bare consistency. A reader who hears only “it’s anomaly-free, like every consistent theory” has missed that the gauge content was not chosen but derived.

Objection 45 (O-BI6-45) (the GGEP skeptic)

BI-6 is presented as the field-strength form of AX-2 (GGEP), via Eq. (16). But equating a modified Bianchi identity with an equivalence principle is a strong claim. Is Eq. (16) a derivation or an analogy?

Response. It is a definitional identification with derived consequences, and the paper should label it as such rather than as a theorem. AX-2 (GGEP) states that gravity and gauge are facets of a single composite connection (Objection 17); the curvature of that composite connection satisfies a Bianchi identity, and Eq. (16) is the statement that the *modified* Bianchi identity $dH_3 = X_4$ is the component of that composite Bianchi identity carrying the mixed gravity-gauge curvature (Section 2). So the link is: GGEP posits the composite connection; the modified Bianchi identity is then a *consequence* of the composite structure, not an independent analogy. Where the objection bites: the step from “composite connection” to “this particular four-component X_4 ” uses the trace normalizations and is a derivation only given the bulk gauge content of (Arisaka, 2026, A2-GG6). So Eq. (16) is a derived identity conditional on the GGEP composite-connection posture, not a free analogy and not an unconditional theorem. The honest phrasing is “BI-6 is what AX-2 becomes at the level of field strengths,” and that is a derivation modulo the composite-connection axiom.

Objection 46 (O-BI6-46) (the model-level critic)

You distinguish a “model-level” anomaly-freedom theorem (Theorem 7.9) from the “constitutional-level” one (Theorem I.4). Doesn’t drawing that distinction concede that the constitutional claim is weaker than an actual model computation?

Response. It concedes that they are different statements, which is a strength rather than a weakness, and Section 7 is explicit about which is which. The model-level theorem (Theorem 7.9) is the concrete computation: take the specific GG-6 bulk spectrum and verify that the anomaly polynomial factorizes and cancels — a calculation anyone can check line by line against Table 18. The constitutional-level theorem (Theorem I.4) is the stronger structural claim: that *any* theory satisfying the constitution has this property, because the constitution forces the spectrum that the model-level computation verifies. The two are related as instance and universal: the model-level result is the existence proof (this spectrum works), the constitutional result is the uniqueness-and-forcing proof (the constitution yields this spectrum and no other). Far from the constitutional claim being weaker, it is the model-level claim plus the derivation of the

spectrum from the axioms. We separate them so a skeptic can accept the checkable model-level computation even while reserving judgment on the constitutional forcing.

Objection 47 (O-BI6-47) (the reproducibility auditor)

The proof rests on trace computations (Table 14, Table 18). Are these independently reproducible from standard group-theory references, or do they rely on cohort-internal conventions an outside reader cannot check?

Response. They are reproducible from standard references, and Appendix D is written so that they can be. The trace data of Table 18 are the quadratic and quartic Dynkin indices of the $SU(5)$ representations carrying the Standard-Model fermions — standard quantities tabulated in the group-theory literature — and the anomaly coefficients of Table 14 are assembled from them by the explicit formulae of Appendix D, with the normalization convention stated at the point of use. The only cohort-internal element is the *choice* of which four blocks to include ($Spin(1,5)$, $SU(5)$, $U(1)_B$, $U(1)_{Q_i}$), which is the constitutional input, not a computational convention. So a reader can reproduce every entry: take the standard $SU(5)$ indices, apply the stated formulae, and recover the matrix; the rank-one collapse (Theorem 5.1) is then a property of that explicitly-built matrix. We regard independent reproducibility of the trace tables as essential and have written the appendix to support it; if any entry cannot be reproduced from standard indices plus the stated conventions, that is a bug to be reported, not a convention to be taken on trust.

Objection 48 (O-BI6-48) (the self-sufficiency auditor)

The paper is ten Parts but defers α , the kaxion mass, the cosmological constant, the loop completion, and the global numerics to five other papers. What does A3 establish *on its own* that does not depend on a companion?

Response. A3's own results, owing nothing downstream, are stated in Section 5.4 and Section 18: (i) the local Green–Schwarz factorization of the BI-6 four-form with a single tensor (Theorem 5.5); (ii) the rank-one collapse of the anomaly matrix and the unique eigenvector $(1, -1, -\frac{2}{5}, -\frac{2}{5})$ (Theorem 5.1); (iii) the trace-ratio identity $2/5 = (L - N)/L$ and its forcing of $(L, N) = (5, 3)$ given minimality (Theorem 8.1); (iv) the gravitational closure without a gravitino (Appendix H); and (v) the global anomaly-freedom theorem on the intended class modulo the checked topological hypotheses (Theorem 6.2). What it borrows or defers — the value of α , the kaxion line, the cosmological constant, the loop and global numerics — it labels as borrowed (Section 4.1). So A3 is the paper that proves the boundary law (the anomaly-free Bianchi identity and its integer consequences); it is neither self-sufficient for the cohort's empirical program nor empty — it establishes the consistency theorem on which the spectrum papers build.

T Empirical exposure and honest status

Cluster: where can A3 lose? The final five objections press on falsifiability and on the honesty of the open-problem ledger.

Objection 49 (O-BI6-49) (the falsification realist)

Every empirical prediction — M_{lock} , the kaxion line, Ω_m , α — is deferred to a companion paper. So A3 itself makes *no* independently falsifiable prediction; it is a pure consistency proof. Is a paper with no own falsifier science, or scaffolding?

Response. A3 makes one sharp, own falsifiable prediction and several structural ones, even though the headline numbers live downstream. The own falsifier is the rank-one structure (Theorem 5.1): the claim that the BI-6 anomaly matrix collapses to rank one with eigenvector $(1, -1, -\frac{2}{5}, -\frac{2}{5})$ is a definite mathematical statement that an independent computation of the SU(5) trace data can confirm or refute (Objection 47), and if the matrix came out rank two the single-tensor construction — and with it AX-4’s admissibility verdict — would fail. The structural predictions include the $n_F = 800$ singlet count (Objection 35), a falsifiable spectral demand, and the trace-ratio 2/5 forcing $(L, N) = (5, 3)$. So A3 is consistency-proof *plus* two checkable structural commitments; it is scaffolding for the empirical papers, but scaffolding that can itself be knocked down by a trace computation. We grant that A3 carries no laboratory falsifier of its own — those are downstream — and that its falsifiability is mathematical rather than experimental.

Objection 50 (O-BI6-50) (the open-problems counter)

You still carry Open Problems O10–O14 as partially resolved or genuinely open. A hostile reader notes that these are the *hard* parts — the general classification (O10), the loop completion (O11), the generation count (O12), the terminated-interval inflow (O13), the decay constant (O14). Is the unresolved residue not the whole problem?

Response. The residue is the hard part, and the paper neither hides this nor accepts that it is “the whole problem.” Table 2 and Section 17 state plainly which results are complete (the local factorization, the rank-one collapse, the trace-ratio forcing, the gravitational closure, the global theorem on the intended class modulo checked hypotheses) and which are open (the full category O10, the loop completion O11, the generation count O12, the interval inflow O13, the decay constant O14). The honest accounting is that the *classical-topological* consistency problem is solved and the *non-perturbative and quantitative-cosmological* problems are open. That is a real and substantial completed core, not a cosmetic one: the rank-one factorization and the trace-ratio lock are the load-bearing results, and they are done. The objection is right that O10–O14 are where the difficulty concentrates, and the paper’s response is to mark them as open with named technology that could close them, rather than to claim them. A framework that labeled its hardest sub-problems “solved” would be the dishonest one; this one labels them open.

Objection 51 (O-BI6-51) (the cosmological-constant realist)

Objection 16 concedes a factor-of-three miss on m_Λ pending a $\zeta_\Lambda \approx 1/50$ correction. But “119 of 120 orders of magnitude recovered” is rhetorical: a factor-of-three error on the one number that matters is still a failed prediction. Why present it as success-adjacent?

Response. The objection is fair, and we state the status here without the rhetorical cushion. The precise situation: the cohort’s leading-order seesaw estimate $m_\Lambda^2 = M_{\text{lock}}^2/M_{\text{Pl}}$ lands within

a factor of about three of the observed $m_\Lambda \approx 2.3$ meV, and the residual factor is attributed to an as-yet-uncomputed zero-mode overlap correction $\zeta_\Lambda \approx 1/50$ ((Arisaka, 2026, A10-Cosmos), Open Problem O6). Two things are true at once: the leading-order estimate is dramatically better than the naive QFT vacuum-energy estimate, *and* a factor-of-three miss is a quantitative failure until the correction is derived. The framing “119 of 120 orders recovered” is legitimate as a statement about the seesaw mechanism’s scale but illegitimate if read as a successful prediction of m_Λ , which it is not yet. The paper’s honest position is therefore: the mechanism is promising, the number is not yet predicted to precision, and the correction factor is an explicit open computation whose success or failure will decide the matter. No wording in this paper should let the factor-of-three read as a hit.

Objection 52 (O-BI6-52) (the kaxion experimentalist)

Once the kaxion mass is pinned to the canonical $8.1 \mu\text{eV}$ (≈ 1.95 GHz), the prediction is a specific haloscope line. If the haloscope program assembling around the band (IBS-CAPP, ADMX-EFR) scans through that frequency and sees nothing, is BI-6 falsified, or does the kaxion mass have an escape hatch?

Response. The answer has two parts, and the dark-matter companion (Arisaka, 2026, C1-Kaxion) states both at their grades. First, the line is not free to slide continuously: its window $\nu_K \in [1.59, 2.13]$ GHz is fenced by the OGN-5 integer $K = 35$ that fixes the anomaly ledger (Section 9), and the seven loadings $\Xi_{\text{cap}} \in \{35, 38, 40, 42, 43, 45, 47\}$ inside it form a ladder that companion grades as an ansatz, pre-registered so that a null across the window counts regardless of the ladder’s later status; the absolute frequency inside the window is a calibration (Open Problem O14), through a bulk scale the companion calibrates on the relic closure rather than derives. Second, what a null kills is therefore the calibrated placement — the benchmark, its ladder, and the inflation-era pinning on which the single-patch history rests — and not the framework: a completed scan of the window with no persistent line at the registered depth ($g_{90} \leq 0.8 g_{\text{DFSZ}}$ at every frequency in it, at the nominal local density with the standard-halo line shape) sends the Kaxion to $\gtrsim 6$ GHz on the unpinned branch or the relic out of this sector, and the companion says so. What *this* paper stakes on the haloscope is the count: the chiral tensor leaves exactly one light pseudoscalar, so a second light QCD-coupled line inside the window, at the first’s grade, refutes the count and names the light mode, and a line whose coupling sits off the DFSZ locus at $> 5\sigma$ falsifies the address the companion derives. The honest caveat is about *sensitivity*: a scan that passes 1.95 GHz without reaching the registered depth is a partial null, not a null, so the falsifier is specified with a coupling threshold, and only a result below that threshold counts. Within that specification the escape hatch the objection worries about is a finite set of pre-registered integers on one side and a declared calibration on the other, and neither can be moved after a null.

Objection 53 (O-BI6-53) (the cohort-coherence auditor)

A3 sits between A2 (the bulk) and A5/A9 (the boundary spectrum). If A2’s bulk geometry or A9’s boundary spectrum is later revised, does the BI-6 anomaly proof

survive, or is it hostage to upstream and downstream changes?

Response. The proof’s dependencies are explicit (the two-way coherence map, Table 21 and Table 22, Section 4.1), and that explicitness is exactly what makes the hostage question answerable rather than alarming. A3 depends on A2 for one thing: the bulk gauge content $\text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Q_i}$ and the product geometry. As long as those are unchanged, the rank-one factorization and the trace-ratio lock are internal to A3 and survive any revision elsewhere. A3 supplies to A9/A5 the anomaly-free Bianchi identity and the integer ledger; if A9’s boundary spectrum were revised, it would have to revise it *consistent with* the anomaly constraint A3 proves, not against it — the constraint flows downstream, so downstream changes are bounded by it rather than threatening it. The genuine exposure is upstream: if A2’s bulk gauge content changed, the four-block anomaly matrix would change and the rank-one result would need re-derivation. That dependency is named in the inward map (Table 22). So A3 is hostage to A2’s gauge content (and says so) but robust against downstream revision, because it is the constraint the downstream must respect.

Objection 54 (O-BI6-54) (the effective-field theorist)

Theorem T-BI6-6.4 prints $\alpha_B^{-1} = 5312/135$ as “exact and rational” while the paper itself concedes, two pages earlier, that $\sum BY = 2/3 \neq 0$ makes $\text{U}(1)_B$ and hypercharge mix kinetically at one loop. In what scheme, at what scale, is the equality exact?

Response. The equality is a tree-level matching condition at the seam, in the basis with no kinetic mixing, evaluated at the idealized Z -attractor values — and the theorem statement now says so. Below the seam, ordinary running and the one-loop mixing proportional to $\sum BY = 2/3$ per generation deform both sides; the mixing enters at $\mathcal{O}(g_B g_Y \sum BY / 16\pi^2)$ per decade and is a percent-level effect over the relevant range, not a change of regime. Under the cohort’s own running correction $\alpha^{-1}(M_Z) = 2^J - N^2/2^J = 127.93$, the share moves from 39.348 to 39.33 and $g_B = 0.565$ is stable at three figures — the quantity the collider reads. Two conditionalities are registered rather than hidden: the proof’s Steps 1–2 consume the capacity identification and stiffness doctrine of the companion papers (T-SM-27.1, T-QUA), and the state whose coupling is derived is the massive tower whose level-by-level inflow consistency is Open Problem O13. Both grades are printed where the theorem is.

Objection 55 (O-BI6-55) (the renormalization skeptic)

Only the products gQ are physical in an Abelian theory; a “derived g_B ” presupposes a charge normalization. Is $\alpha_B^{-1} = (2/5)\alpha_Y^{-1}$ a prediction or a convention?

Response. It would be a convention if the charges were floating; they are not. The charge normalization is fixed once and for all by the integer baryon ledger — quarks at $B = 1/3$, component-resolved on the split spectrum — and POS-6 forbids re-scaling it. With Q pinned to integers (times $1/3$), the products $g_B Q$ that a collider measures determine g_B with no residual freedom, and the theorem’s content is a statement about those measurable products: the baryonic seam’s stiffness is its ledger share of the cell’s. The same discipline is what makes $\alpha^{-1}(0) = 2^J + N^2$ a claim rather than a choice of units.

Objection 56 (O-BI6-56) (the heterotic string theorist)

Earlier versions of this Part claimed 10D anomaly cancellation requires supersymmetry. The $O(16) \times O(16)$ heterotic string has canceled its anomalies without supersymmetry since 1986. What, then, survives of the contrast?

Response. The claim was wrong and is repaired in this version; the counterexample is real and forty years old. What survives is the contrast that mattered all along: what fixes the spectrum. The non-supersymmetric 10D strings obtain their matter content from a worldsheet construction, and the supersymmetric 6D landscape locks its spectra through multiplet structure; BI-6 is non-supersymmetric *and* its reservoir is fixed by a Diophantine system descending from constitutional clauses, with zero continuous dials. The comparison Part now states the supersymmetry facts as they are and locates the genuine difference in the spectrum's provenance.

Objection 57 (O-BI6-57) (the Kaluza–Klein theorist)

The companion paper prints an angular tower at $M_\ell \sim |\ell|E_{\text{GUT}}$. A tower spacing is a radius being measured. How is that consistent with the doctrine that the circle carries no radius?

Response. The doctrine, stated with the care this objection forces, is that the radius is *frozen*, *not absent*: the circle's circumference is fixed at $\sim 1/E_{\text{GUT}}$ by the intended-class background data, POS-6 removes the radion as a propagating four-dimensional mode, and below E_{GUT} the circle registers only through its holonomy — which is why every low-energy θ -dependent quantity in the framework is a holonomy or a winding number. The angular tower at $|\ell|E_{\text{GUT}}$ is the one place the frozen value would register, twelve orders of magnitude above the lock cluster. What the framework forbids is not the tower but the modulus: there is no flat direction whose expectation value the tower spacing tracks.

Objection 58 (O-BI6-58) (the general relativist)

POS-6 freezes $g_{\theta\theta}$ and the product structure. A frozen metric component breaks six-dimensional diffeomorphism invariance — so what symmetry is your gravitational anomaly budget, with its careful p_2 bookkeeping, actually for?

Response. For the gauge redundancy, which is intact. The anomaly analysis of Part II is a statement about local frame and diffeomorphism *redundancy* — the covariance whose gauging is what $\text{Spin}(1,5)$'s appearance in G_{Bulk} means — and that redundancy is not touched by POS-6. What POS-6 fixes is background data: the product split, the frozen structural moduli, the mode content of the fluctuation spectrum. A background that breaks isometries breaks no gauge symmetry — exactly as an interval compactification's boundary conditions reduce the isometry group while the diffeomorphism redundancy, and its anomaly obligations, survive untouched. The p_2 bookkeeping is therefore an obligation the theory genuinely has, and the completion sector that discharges it is load-bearing, not optional.

Objection 59 (O-BI6-59) (the swampland theorist)

Your completion sector is $4\Omega - 1$ free spectator tensors and 32Ω singlet fermions. Free tensors carry exact continuous higher-form global symmetries, and a decoupled

singlet sector carries an enormous flavor symmetry — both are precisely what the no-global-symmetries conjecture forbids.

Response. Conceded as an open exposure, and the swampland section now grades it so. The gauge sector is clean — all four factors gauged, no ungauged continuous structural symmetry. The completion sector is not yet clean: in the minimal action its spectator tensors carry higher-form symmetries and its singlets a flavor symmetry, and the framework has not exhibited the couplings, bulk masses, or intended-class boundary conditions that lift them. Three candidate resolutions are on the table — gravitational couplings alone (present, but weak), Dirichlet data of the admissible class (which does break higher-form symmetries on an interval), and boundary projection — and none is computed. Until one is, the honest grade for the no-global-symmetries conjecture is structural compatibility for the gauge sector with the completion sector deferred; a proof that no lifting exists would be evidence against the completion sector as constructed, and we register that exposure.

Closing remark on the objections. After all, an honest framework should know which questions a working physicist is most likely to raise on first reading, and should answer those questions before they become obstacles to engagement. We have addressed fifty-nine such objections above, grouped into eleven clusters: the anomaly proof and the trace ratio (Objections 1–3), global anomaly freedom and the non-supersymmetric setting (4–7), minimality and the parsimony axiom (8–12), convergences and empirical predictions (13–17), the local descent and the four-form X_4 (18–24), the global completion via Dai–Freed and cobordism (25–30), the OGN-5 ledger as a theorem (31–36), the relationship to string theory (37–43), the constitutional framing and self-sufficiency (44–48), empirical exposure and honest status (49–53), and the second-round audit of the derived coupling, the terminated circle, and the swampland grading (54–59). Each is the framework’s standing offer to be wrong, in detail, in a specific way; the deeper technical comparisons live in Sections 10–12 and the Open Problems triage in Section 17.

Author’s Note

The author of the GG-6 program is an experimental physicist—not a string theorist, not a formal quantum field theorist, and not a mathematician. The GG-Theory Constitution and its consequences were developed from the perspective of an experimentalist who studies complex systems across multiple scientific domains, from particle physics to neuroscience. The framework is driven by the question *what must any admissible theory satisfy?* rather than by the question *what does a specific UV-complete theory predict?*

This background has both advantages and limitations. The advantage is a willingness to question structural assumptions (such as the necessity of supersymmetry or the primacy of Planck-scale completion) that a theorist trained within the string tradition might take as given. The limitation is the absence of deep expertise in the most technical aspects of the subject—*anomaly polynomials, modular forms, bordism theory, derived categories*—that the string community has spent decades developing.

The present paper has attempted to be mathematically rigorous within its scope, but not every

subtlety may have been addressed. The author is confident in the structural proof (Section 5), the model-level benchmark (Section 7), the OGN-5 determination (Sections 8–9), and the comparisons (Sections 10–12). The author is aware that the global anomaly technology (Dierigl and Tartaglia, 2025, Monnier and Moore, 2019), the charge-lattice refinements (Seiberg and Taylor, 2011), and the bordism-theoretic constraints (Hopkins and Singer, 2005) may impose additional conditions that the present analysis has not fully explored.

Collaboration with the string theory community would be most valuable. The Minimal BI-6 Spectrum Problem (entry O8 of Section 17) is a well-posed mathematical question that the community’s expertise in 6D anomaly polynomials and index theory is ideally suited to address. The global consistency checks that Monnier and Moore developed for $\mathcal{N} = (1, 0)$ SUGRA could be applied to the non-SUSY GG-6 setting to test whether additional obstructions arise. The enumeration technology of Hamada and Loges (Hamada and Loges, 2024) could be adapted to the non-SUSY Diophantine system.

The author therefore extends an open invitation: *if the structural proof is correct, what are its consequences? If it has a gap, where exactly is it?* Either answer would advance the field.

A final word to graduate students and early-career theorists. The tools developed by the string theory community over the past four decades—anomaly polynomials, descent equations, index theory, bordism, holographic renormalization, charge-lattice quantization—are among the most powerful mathematical instruments in theoretical physics. These tools are not tied to any single interpretive framework. They can be applied to supersymmetric string compactifications, to non-supersymmetric field theories, to condensed-matter systems, and—as the present paper attempts to show—to axiomatic approaches that begin from different premises. A young theorist who masters these tools will find no shortage of problems to solve, regardless of which theoretical direction ultimately prevails. The questions raised by this paper—the Minimal BI-6 Spectrum Problem, the extension of the global anomaly theorem beyond the intended non-SUSY GG-6 category, the connection between OGN-5 arithmetic and F-theory lattice conditions, the operational meaning of the 6D→4D channel—are concrete, well-posed mathematical problems that can form the basis of productive research programs. The author would welcome collaboration with researchers at any career stage.

What the objections established. Answering fifty-nine objections in sequence changes a paper, and it is worth recording how. Several of them could not be answered as posed, and the responses say so: the trace ratio is not fixed by anomaly freedom alone; the anomaly test is passed by a family of Abelian coefficients rather than by one; the explicit mapping-torus partition functions have not been computed; the bordism group has not been evaluated in closed form; the generation count is not derived. Each of those concessions is now carried in the main text as well, in the section that makes the claim, rather than living only here where a hurried reader would miss it.

That is the use of an objections Part. It is not a defensive exercise and it is not a summary. It is the place where the strongest available criticism is put in writing and the paper is made to answer it, and a framework that comes through the process unchanged has not taken the process seriously. This one did not come through unchanged.

Part X

References

Cited works, each with a relevance note linking it to the present paper

Unlike a conventional reference list, every entry below carries a short *Relevance* note: a sentence or two saying what the cited work contributes and where in the present paper it bears. The aim is to turn the bibliography from a bare list of sources into a navigable companion — a reader can move through the references and see, at a glance, which theorem, proof, or section each one underwrites. The notes are written for two readers at once. For the newcomer, they trace the intellectual lineage of the anomaly proof: from the foundational work on the Green–Schwarz mechanism, differential cohomology, and bordism invariance, through to the twenty-two-paper GG-Theory cohort in which the present paper supplies the local-and-global anomaly theorem. For the specialist, they make the paper’s debts explicit and checkable — each note names the specific result it supports, so that no citation is merely decorative.

The references are organized in two groups — first the author’s GG-Theory papers (Section A below), then all other works (Section B), each in alphabetical order by first author. The cohort papers (GG-A1-Constitution–GG-A11-Origin, with the C- and P-series) are cited under the canonical keys fixed by the cross-paper coherence map of the GG-Theory program.

Each entry carries a note saying what the work established and where the present paper uses it. The notes exist because a bare citation list records that a debt was incurred without recording what it was for, and in a paper that borrows its central technique wholesale from another tradition that distinction is worth the pages it costs. A reader who wants to know how much of this construction is new can read the notes and find out.

A. Papers by Katsushi Arisaka (the GG-Theory cohort)

Arisaka, K. (2026). *The GG-Constitution: Four Axioms and Seven Postulates of the GG-Theory Program, From Which GG-6, BI-6, OGN-5, and GDOS Descend as Theorems*, Arisaka-GG-A1-Constitution.

Preprint, available at <https://preprints.arisaka-gg.org/a1-constitution/>.
[doi:10.5281/zenodo.22063867](https://doi.org/10.5281/zenodo.22063867).

Relevance. The constitutional charter of GG-Theory. States the four foundational axioms (LC, GGEP, OGEP, MSMF) and the seven postulates that serve as the sole admissibility input for the present anomaly proof. Every theorem of A3-BI6 — from the structural proof of BI-6 cancellation in §5 to the constitutional reading of §12 — traces its primitive input to one or more clauses of this charter; the constitutional placement of BI-6 on Stratum 2 in Section 12 is the most explicit invocation. First cited in the Executive Summary.

Arisaka, K. (2026). *GG-Theory with GG-6: Geometry–Gauge Locking at the 6D Bulk (GG-6) at the Heart of Grand Geometric Theory (GG-Theory)*, Arisaka-GG-A2-GG6.

Preprint, available at <https://preprints.arisaka-gg.org/a2-gg6/>.

doi:10.5281/zenodo.22064298.

Relevance. Derives, from the GG-Theory Constitution, the six-dimensional bulk $M_4 \times I_r \times S_\theta^1$, the four-component composite gauge group $\text{Spin}(1, 5) \times \text{SU}(5) \times \text{U}(1)_B \times \text{U}(1)_{Qi}$, the modified Bianchi identity BI-6, and the OGN-5 integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$. A3-BI6 is the rigorous local-plus-global anomaly proof for that construction; we cite GG-A2-GG6 throughout §1.3, §5.1, and the OGN-5 derivation in §9. First cited in the Executive Summary.

Arisaka, K. (2026). *GDOS: Geometric Dynamics of Open Systems — 6D GG-6 Bulk Projected to 4D Boundary for Quantum Measurement, General Relativity, and QFT*, Arisaka-GG-A5-GDOS.

Preprint, available at <https://preprints.arisaka-gg.org/a5-gdos/>.

doi:10.5281/zenodo.22064304.

Relevance. Extends the bulk GG-6 framework to the boundary through the OGGEF channel as a completely positive trace-preserving map and derives 4D quantum mechanics, general relativity, and the low-energy Standard Model as boundary projections. The local and global anomaly freedom of BI-6 proved in A3-BI6 is the lawfulness prerequisite for that boundary channel. We cite GG-A5-GDOS in §1.5 for the forward link from the bulk anomaly theorem to the boundary law, and in the Scope-and-Limits paragraphs of §17. First cited in §1.

Arisaka, K. (2026). *The Qi-4 Pillars — Qi-QUA, Qi-EP, Qi-CON, Qi-HAL: The Quantum-Information Backbone of GDOS*, Arisaka-GG-A7-Qi4.

Preprint, available at <https://preprints.arisaka-gg.org/a7-qi4/>.

doi:10.5281/zenodo.22064308.

Relevance. Establishes the four quantum-information pillars (Qi-QUA, Qi-EP, Qi-CON, Qi-HAL) on the boundary GDOS runtime and calibrates the bioenergetic conversion factor $Q_B \approx 0.1555$ eV per qubit against ATP hydrolysis at biological temperatures. These boundary Qi-4 pillars require $\text{U}(1)_{Qi}$ as host symmetry — one of the three empirical reasons that A3-BI6 adopts the four-component bulk in Theorem 6.22 and §6.6, rather than the three-component option also admitted by the constitution. First cited in §4.1.

Arisaka, K. (2026). *The Standard Model as Theorem of GG-Theory: Its 28 Parameters as Observations of One Six-Dimensional Geometry*, Arisaka-GG-A9-SM.

Preprint, available at <https://preprints.arisaka-gg.org/a9-sm/>.

doi:10.5281/zenodo.22064312.

Relevance. Derives the 28 continuous Standard-Model parameters — the three gauge couplings, charged-lepton and quark masses, CKM and PMNS angles, Higgs sector, QCD theta angle, and cosmological constant — from the OGN-5 integer ledger and the 6D bulk. A3-BI6 provides the upstream anomaly-freedom certificate without which that derivation cannot be deployed; we cite GG-A9-SM at the empirical-prediction paragraphs of §14 and at the cross-paper coherence map in §16. First cited in §1.

Arisaka, K. (2026). *The Standard Cosmology as Theorem of GG-Theory: The Cosmic Inventory as Observations of One Six-Dimensional Geometry*, Arisaka-GG-A10-Cosmos.

Preprint, available at <https://preprints.arisaka-gg.org/a10-cosmos/>.

doi:10.5281/zenodo.22064314.

Relevance. Demonstrates that the Green–Schwarz normalization Ω is driven to $\Omega = 1$ as a cosmological attractor by the GUT-adjacent slow smooth inflationary descent, and predicts kaxion dark matter at $8.1 \mu\text{eV}$ plus Kaluza–Klein resonances at $M_{\text{lock}} \approx 3.85 \text{ TeV}$. Both predictions presuppose the local + global anomaly-freedom certificate of BI-6 established in A3-BI6. We cite GG-A10-Cosmos in the Green–Schwarz-normalization discussion of §14 and in the cross-paper coherence map. First cited in §4.1.

Arisaka, K. (2026). *Anomaly Freedom of BI-6: The Modified Bianchi Identity (BI-6) for the Six-Force, Six-Dimensional Bulk (GG-6)*, Arisaka-GG-A3-BI6.

Preprint, available at <https://preprints.arisaka-gg.org/a3-bi6/>.

doi:10.5281/zenodo.22064300.

Relevance. The present paper; listed for completeness as the GG-A3-BI6 node of the cohort map (§4.1). First cited in §4.1.

Arisaka, K. (2026). *The GRIP on GDOS — GRIP: the Geometric Engine of Symmetry Breaking, from the Big Bang to Mind*, Arisaka-GG-A6-GRIP.

Preprint, available at <https://preprints.arisaka-gg.org/a6-grip/>.

doi:10.5281/zenodo.22064306.

Relevance. The geometric Gi-4 pillars of the open-system boundary law; the geometric sibling of the Qi-4 pillars of GG-A7-Qi4. Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *Quantum Mechanics Derived from GG-Theory as Projection from 6D Bulk to 4D Boundary*, Arisaka-GG-A8-QM.

Preprint, available at <https://preprints.arisaka-gg.org/a8-qm/>.

doi:10.5281/zenodo.22064310.

Relevance. Recasts textbook quantum mechanics in the language of GG-Theory via GDOS and OGGEF; the QM stratum of the Consequence Trio. Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *The Geometric Inevitability of Cosmos, Life, and Mind: The Self-Referential Closure of the GG-Theory Program*, Arisaka-GG-A11-Origin.

Preprint, available at <https://preprints.arisaka-gg.org/a11-origin/>.

doi:10.5281/zenodo.22064319. No result of the present paper depends on it.

Relevance. The closing capstone of the cohort (recursive uniqueness; the cosmos–life–mind chain). Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *The GG-6 Lock Cluster: A Non-Spatial Kaluza–Klein Resonance near 3.85 TeV as the Standing LHC Falsifier of GG-Theory*, Arisaka-GG-P1-KK.

Preprint, available at <https://preprints.arisaka-gg.org/p1-kk/>.

doi:10.5281/zenodo.22064321.

Relevance. Predicts the Kaluza–Klein lock cluster near $M_{\text{lock}} \approx 3.85$ TeV. Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *Why Three Generations: A Three-Leg Derivation of $N_g = 3$ from BI-6, MSMF, and GDOS Morse Stability*, Arisaka-GG-P2-N3.

Preprint, available at <https://preprints.arisaka-gg.org/p2-n3/>.
doi:10.5281/zenodo.22064323.

Relevance. Derives the three-generation count $N_g = 3$, building directly on the anomaly proof of the present paper. Its Appendix H is the inflow computation that consumes this paper’s primitive lift, doubling construction, unit reservoir and $a = 1$ refinement as bulk-side inputs: it pushes the BI-6 differential data forward through the Hopkins–Singer quadratic functor on the benchmark background of the intended class and proves the value is the boundary’s Berry line bundle. Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *The Electron as the Ground State of Nature: The Six Oldest Questions about the Electron, Closed and Fenced within GG-Theory*, Arisaka-GG-P3-Electron.

Preprint, available at <https://preprints.arisaka-gg.org/p3-electron/>.
doi:10.5281/zenodo.22064325.

Relevance. Treats the electron as the ground state of nature within GG-Theory. Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *The Real Geometry Beneath the Complex Numbers of Physics: Positive Frequency is Holomorphy — from Cardano’s Cubics to Twistor Space, via the Sixth Dimension of GG-Theory*, Arisaka-GG-P4-Real.

Preprint, available at <https://preprints.arisaka-gg.org/p4-real/>.
doi:10.5281/zenodo.22064327.

Relevance. The origin-of- i member paper of the GG-Theory program (Foundations Branch): derives the single imaginary unit of boundary physics from the two real structures of the GG-A2-GG6 arena (the compact phase circle and the causal clock). On the twenty-two-paper cohort map it stands on the spine’s left flank, fed by GG-A2-GG6’s Theorem-0 arena and feeding the quantum-mechanical recast of GG-A8-QM; the present paper cites it at roster level. Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *The Mirrors of Matter: Why Every Particle Has a Twin, Why Nature’s Mirrors Break, and Why the Universe Is Made of Matter*, Arisaka-GG-P5-CPT.

Preprint, available at <https://preprints.arisaka-gg.org/p5-cpt/>.
doi:10.5281/zenodo.22064329.

Relevance. The CPT member paper of the GG-Theory program (Foundations Branch): reads C, P, and T as the orientation reversals of the three real bulk factors and carries the matter–antimatter door. That orientation reading of the discrete symmetries makes it an independent consumer of the anomaly-inflow logic proved here; the present paper cites it at roster level. Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *The Proton as the Ground Geometric Attractor of Matter: The Eleven Questions of the Proton, Closed and Fenced within GG-Theory*. Arisaka-GG-P6-Proton preprint. Preprint, available at <https://preprints.arisaka-gg.org/p6-proton/>. doi:10.5281/zenodo.22064331.

Relevance. The fourth member of the Predictions Branch, sister to GG-P3-Electron: the proton as the GGA of the baryonic stratum, with baryon number carried as a winding of the compact geometry rather than as a bookkeeping label — written by the same twist that breaks the unified gauge symmetry and splits the Higgs multiplet — so that the decay the grand unified theories insisted on is forbidden by the geometry rather than merely unobserved. Eleven questions closed as faces of one fact, every import at its proved grade and every kill criterion stated; a single verified proton decay falsifies it. First cited in §4.1.

Arisaka, K. (2026). *The Kaxion: Cold Dark Matter as the Non-Spatial Compact-Angle Pseudoscalar of GG-6*, Arisaka-GG-C1-Kaxion. Preprint, available at <https://preprints.arisaka-gg.org/c1-kaxion/>. doi:10.5281/zenodo.22064333.

Relevance. The kaxion dark-matter candidate near $8.1 \mu\text{eV}$: a single line in a fenced window. Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *The Hubble Tension in the Light of GG-Theory: The Hubble Constant as One Reading of a Derived Expansion History $H(t)$* , Arisaka-GG-C2-Hubble. Preprint, available at <https://preprints.arisaka-gg.org/c2-hubble/>. doi:10.5281/zenodo.22064336.

Relevance. Reads the Hubble tension as the two ends of one six-dimensional geometric corridor. Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *Black-Hole Entropy under GG-Theory: The GDOS Reading of the Horizon as an OGGE Port — with the Exact Bekenstein–Hawking $A/4G_N$ Derived from Induced Gravity*, Arisaka-GG-C3-BH. Preprint, available at <https://preprints.arisaka-gg.org/c3-bh/>. doi:10.5281/zenodo.22064338.

Relevance. Black-hole entropy under GG-Theory (horizon as an OGGE port; exact Bekenstein–Hawking $A/4$). Cited in the cohort overview, §4.1. First cited in §4.1.

Arisaka, K. (2026). *Symmetry, Conservation, and Geometry: Seven Steps from Noether, Bianchi, to Hopkins–Singer, and the Ledger the Open-System Turn Pays Out*. Arisaka-GG-W1-Symmetry preprint. Preprint, available at <https://preprints.arisaka-gg.org/w1-symmetry/>. doi:10.5281/zenodo.22064340.

Relevance. The first of the two white papers, and the *diachronic* one: it walks the century of symmetry and conservation in the order it was found and shows the conviction that symmetry causes conservation loosening at seven historical steps, until what remains conserved is an integer with nowhere to move — then places the program at the arc’s end, at declared grades, with its

payout set beside the measurements. Derives nothing new: every result is carried at its parent's declared grade. First cited in §4.1.

Arisaka, K. (2026). *The Circle That Remembers, the Interval That Forgets: GG-Theory and the Sixteen Problems of Particle Physics and Cosmology*. Arisaka-GG-W2-Holonomy preprint.

Preprint, available at <https://preprints.arisaka-gg.org/w2-holonomy/>.

doi:10.5281/zenodo.22064342.

Relevance. The second of the two white papers, and the *synchronic* one: it takes a single exact statement — a holonomy requires a closed loop and the interval fiber has none, so the circle remembers and the interval forgets — and follows it to sixteen of the questions physics has not closed, closing with the named, dated measurements on which six of the sixteen treatments die outright. Like its sibling it derives nothing new, carrying every result at its parent's declared grade. First cited in §4.1.

Arisaka, K. (2026). *From Euclid to GG-6: The Realization of Euclid's Elements in Six Dimensions — GU-5, OGN-5, and a Parameter-Free Reconstruction of Modern Physics* Arisaka-GG-A4-Euclid, UCLA preprint.

Preprint, available at <https://preprints.arisaka-gg.org/a4-euclid/>.

doi:10.5281/zenodo.22064302.

Relevance. GG-A4-Euclid is the consequence paper of the foundation block, completing the partial-geometrization program that runs Maxwell 1865, Minkowski 1908, Einstein 1915, Kaluza–Klein 1919–26, Wilson–Hosotani 1974–83, Randall–Sundrum 1999, and the Green–Schwarz mechanism 1984. We cite it in Section 2 as the historical positioning paper that frames the 1984 GS achievement as the moment higher-dimensional gauge theory glimpsed AX-2 (GGEP) at the technical level. First cited in §2.

B. Other references

S. Adler. *Axial-vector vertex in spinor electrodynamics*, *Phys. Rev.* **177** (1969) 2426.

Relevance. Discovers the axial-vector anomaly in spinor electrodynamics: the one-loop quantum failure of classical $\partial_\mu j_5^\mu = 0$ that became the prototype for every higher-dimensional anomaly that BI-6 must cancel. In A3-BI6 we cite Adler at the historical motivation in §1 and again in §3 when the descent formalism is introduced; the entire eight-part proof obligation of §5.4 exists because this 1969 result made gauge-anomaly cancellation a non-negotiable consistency requirement. First cited in the Executive Summary.

L. Alvarez-Gaumé and E. Witten. *Gravitational anomalies*, *Nucl. Phys. B* **234** (1984) 269.

Relevance. Establishes the modern index-theoretic classification of gravitational and mixed gauge-gravity anomalies in even dimensions: the framework in which one writes down anomaly polynomials, identifies factorization conditions, and reads off the resulting cancellation requirements. A3-BI6 uses this framework throughout the structural proof in §5 and the explicit anomaly-matrix

computation of Appendix H; the four-component coefficient vector $(a, b, c, d) = (1, -1, -2/5, -2/5)$ is fixed by exactly the index data that this paper systematized. First cited in §5.

L. Alvarez-Gaumé, P. H. Ginsparg, G. W. Moore, and C. Vafa. *An $O(16) \times O(16)$ heterotic string*, *Phys. Lett. B* **171** (1986) 155–162.

Relevance. Constructs the tachyon-free non-supersymmetric $O(16) \times O(16)$ heterotic string, whose ten-dimensional anomalies cancel by the Green–Schwarz mechanism with no gravitino and no superpartners — the standing counterexample to the claim that 10D anomaly cancellation requires supersymmetry. Cited at the supersymmetry contrast of §10 and in Objection O-BI6-56, together with (Dixon and Harvey, 1986), which obtained the same theory independently.

M. F. Atiyah, V. K. Patodi, and I. M. Singer. *Spectral asymmetry and Riemannian geometry. I*, *Math. Proc. Camb. Phil. Soc.* **77** (1975) 43–69.

Relevance. Introduces the eta invariant and the index theorem for manifolds with boundary that the global half of this paper’s argument runs on. The Dai–Freed formulation of a global anomaly is an exponentiated eta invariant, and the per-species factor $\exp(2\pi i \cdot \text{index})$ evaluated in Section 6 is this theorem applied to the benchmark spectrum. First cited in Section 6.

I. Basile and G. Leone. *Anomaly constraints for heterotic strings and supergravity in six dimensions*, *JHEP* **2404** (2024) 067, arXiv:2310.20480.

Relevance. Establishes anomaly constraints for heterotic and non-SUSY supergravity constructions in six dimensions. Directly relevant to the non-SUSY heterotic boundary of the 6D landscape that A3-BI6 must engage with. We cite Basile–Leone at the swampland and 6D-classification paragraphs of §11; their analysis shows that anomaly cancellation continues to function as the principal admissibility filter once supersymmetry is removed — exactly the regime in which the four-component BI-6 of A3-BI6 operates. First cited in §11.

Z. K. Baykara, Y. Hamada, H.-C. Tarazi, and C. Vafa. *On the string landscape without hypermultiplets*, *JHEP* **2404** (2024) 121, arXiv:2309.15152.

Relevance. Constructs 6D string vacua without hypermultiplets via asymmetric orbifolds, pushing the boundary of 6D string constructions toward non-standard configurations. In A3-BI6 we cite Baykara–Hamada–Tarazi–Vafa at the non-geometric-compactification paragraphs of §11; the comparison highlights that even within the string-theoretic tradition, the assumed matter content can be relaxed — a flexibility that BI-6 achieves constitutionally through the OGN-5 integer ledger of §9 rather than through orbifold engineering. First cited in §11.

C. Becchi, A. Rouet, and R. Stora. *Renormalization of gauge theories*, *Ann. Phys.* **98** (1976) 287.

Relevance. Establishes the BRST formalism: the cohomological language in which anomaly cancellation, Slavnov–Taylor identities, and the Wess–Zumino consistency condition acquire their natural algebraic form. A3-BI6 uses BRST language throughout the descent proof of §5 and the explicit BRST cancellation in Appendix B, where the seven-form, six-form, and one-loop variation are written as BRST cocycles whose exact cancellation by the Green–Schwarz counterterm closes the local anomaly ledger of BI-6. First cited in §2.

K. Becker, A. Kehagias, E. Sezgin, D. Tennyson, and A. Violaris. *New anomaly-free supergravities in six dimensions*, *JHEP* **2405** (2024) 144, arXiv:2311.03337.

Relevance. Discovers new classes of anomaly-free 6D supergravity models, demonstrating that the 6D landscape is still actively being explored within the SUSY framework. In A3-BI6 we cite Becker–Kehagias-Sezgin-Tennyson-Violaris at the existing-literature paragraphs of §11; the comparison reinforces the contrast with the constitutional approach of GG-6, where the four-component coefficient vector (a, b, c, d) of BI-6 is fixed uniquely by the trace-ratio theorem of §9 rather than emerging as one possibility among many. First cited in §11.

K. Becker, M. Becker, and J. H. Schwarz. *String Theory and M-Theory: A Modern Introduction*, Cambridge University Press (2007).

Relevance. The modern graduate string-theory text; its Ch. 5 treatment of anomaly cancellation is the recommended companion to Appendix C. First cited in the Part VII course introduction.

J. S. Bell and R. Jackiw. *A PCAC puzzle: $\pi^0 \rightarrow \gamma\gamma$ in the sigma model*, *Nuovo Cimento A* **60** (1969) 47.

Relevance. Independent discovery of the chiral anomaly through the $\pi^0 \rightarrow \gamma\gamma$ amplitude; together with Adler this established that quantum anomalies are physical and unavoidable rather than a calculational artifact. In A3-BI6 we cite Bell–Jackiw alongside Adler in §1 and at the motivation for the descent formalism in §3; the Wess–Zumino consistency condition implemented in Appendix B is the cohomological generalization of the Adler–Bell–Jackiw obstruction to the six-dimensional setting. First cited in the Executive Summary.

K. Benakli, M. D. Goodsell, and A. Goudeau. *An anomaly-free baryonic completion of the Standard Model*, arXiv:2607.12009 [hep-ph] (2026).

Relevance. The orthodox four-dimensional answer to the question BI-6 answers geometrically: baryon number gauged directly, with the anomaly canceled by a secluded chiral sector and the Z' Higgsed by a singlet scalar. Locally and globally anomaly-free on its own class of backgrounds; every difference with the present construction is structural. Engaged in full in Subsection 11.6: the added-matter/inflow fork, the scanned-versus-derived parameters, the 3.85 TeV coincidence of their non-unified benchmark with M_{lock} , and dark sectors sixteen to seventeen orders of magnitude apart. First cited in the Executive Summary.

R. A. Bertlmann. *Anomalies in Quantum Field Theory*, Oxford University Press (1996).

Relevance. The book-length cohomological treatment of anomalies — descent equations, BRST cohomology, and the Stora–Zumino chain — matching the route taken in Appendix B. First cited in the Part VII course introduction.

L. Bianchi. *Sui simboli a quattro indici e sulla curvatura di Riemann*, *Atti della Reale Accademia dei Lincei, Rendiconti* **11** (1902) 3–7.

Relevance. Establishes the differential identity for the Riemann curvature tensor whose generalization to gauge connections, $dF = 0$, is the classical Bianchi identity that the modified BI-6 of GG-Theory generalizes in the Green–Schwarz manner. In A3-BI6 we cite Bianchi 1902 in the historical motivation of §1 and at the introduction of the modified identity in §5.1; the

modification $dH_3 = X_4 \neq 0$ that defines BI-6 is the topological completion of Bianchi’s classical identity. First cited in the Executive Summary.

A. Bilal. *Lectures on anomalies*, arXiv:0802.0634.

Relevance. The standard modern lecture notes on anomalies — triangle to descent to inflow — at exactly the level of the course in Part VII; recommended first reading beside Appendices B and H. First cited in the Part VII course introduction.

R. Bott. *The stable homotopy of the classical groups*, *Annals of Mathematics* **70** (1959) 313–337.

Relevance. Supplies the stable-homotopy facts used in the automatic-admissibility theorem, in particular the calculation of the relevant π_k groups for $SU(5)$ that control the global obstructions. In A3-BI6 we cite Bott in Appendix I at the homotopy-group computation showing that the Postnikov 4-type of $SU(5)$ is $K(\mathbb{Z}, 3)$, with $\pi_1 = \pi_2 = \pi_4 = 0$ and $\pi_3(SU(5)) = \mathbb{Z}$, which is the input needed to make the global anomaly vanish automatically. First cited in §I.

R. Bousso and J. Polchinski. *Quantization of four-form fluxes and dynamical neutralization of the cosmological constant*, *JHEP* **0006** (2000) 006, arXiv:hep-th/0004134.

Relevance. Introduces the multi-flux mechanism for dynamical neutralization of the cosmological constant, from which the discretuum of vacua follows; the widely quoted count of order 10^{500} is due to Douglas (2003), not to this paper. Cited here as the canonical reference for the landscape problem that GG-6’s finite Diophantine approach aims to circumvent. Bousso–Polchinski is listed here as background for the finite-versus-infinite landscape discussion of §10 and §11, where the unique OGN-5 solution stands in pointed contrast. First cited in §6.

T. D. Brennan, F. Carta, and C. Vafa. *The string landscape, the swampland, and the missing corner*, arXiv:1711.00864.

Relevance. Comprehensive review of the swampland conjectures: the systematic articulation of what distinguishes effective theories with UV string completions from those without. In A3-BI6 we cite Brennan–Carta–Vafa at the swampland paragraphs of §11; the comparison frames the constitutional question for GG-6 — whether BI-6 sits in the string landscape, the swampland, or a third category defined by the GG-Theory Constitution rather than by string consistency. First cited in §11.

J.-L. Brylinski. *Loop Spaces, Characteristic Classes and Geometric Quantization*, Birkhäuser (1993).

Relevance. Standard reference for gerbes, the Dixmier–Douady class, and degree-three integral cohomology; underwrites the gerbe / differential-cohomology reading of the Green–Schwarz tensor in §5.2. First cited in §5.

C. Bustamante, Y. Chemla, N. Forde, and D. Izhaky. *Mechanical processes in biochemistry*, *Annu. Rev. Biochem.* **73** (2004) 705–748.

Relevance. Standard reference for the mechanical biochemistry of ATP hydrolysis at biological temperatures: supplies the empirical anchor for ATP hydrolysis (standard free energy 30.5 ± 0.5 kJ/mol) against which the Arizaka-equation quantum $Q_B \approx 0.1555$ eV per qubit is calibrated,

one phosphoanhydride bond carrying two quanta ($2 N_A Q_B = 30.0$ kJ/mol). In A3-BI6 we cite Bustamante at the bioenergetic-anchor paragraph of §1 and at Objection 3 of §IX, where the four-component bulk’s adoption of $U(1)_{Q_i}$ is justified on three empirical grounds, this being the first. First cited in the Executive Summary.

C. G. Callan and J. A. Harvey. *Anomalies and fermion zero modes on strings and domain walls*, *Nucl. Phys. B* **250** (1985) 427.

Relevance. Establishes the anomaly-inflow mechanism on domain walls and strings: the foundational result that bulk topological terms can carry chiral-anomaly inflow to boundary defects. Callan–Harvey inflow is the conceptual ancestor of the OGGEp boundary channel that GG-A5-GDOS implements as a CPTP map; listed here as essential background reading for the constitutional reading of BI-6 as the bulk anomaly mechanism whose boundary projection underlies GDOS. First cited in §5.

M. Carena, E. Pontón, T. M. P. Tait, and C. E. M. Wagner. *Opaque branes in warped backgrounds*, *Phys. Rev. D* **67** (2003) 096006, arXiv:hep-ph/0212307.

Relevance. Brane-localized gauge kinetic terms in a warped background: the mechanism by which a kinetic term on the infrared end makes the first Kaluza–Klein gauge mode light and repels the tower’s wavefunctions from the boundary, lowering its contribution to the oblique parameters. Cited at Open Problem O15, where the same term on the terminated seam is the one escape from GG-A9-SM’s oblique tension that the constitution can decide. First cited in §17.

S.-S. Chern and J. Simons. *Characteristic forms and geometric invariants*, *Ann. Math.* **99** (1974) 48.

Relevance. Introduces the Chern–Simons transgression three-form ω_3 with $d\omega_3 = \text{tr } F^2$, the local primitive that makes the modified Bianchi identity $dH_3 = X_4$ exact (Step 3 of §5.2). First cited in §5.

H. Davoudiasl, J. L. Hewett, and T. G. Rizzo. *Brane-localized kinetic terms in the Randall–Sundrum model*, *Phys. Rev. D* **68** (2003) 045002, arXiv:hep-ph/0212279.

Relevance. The companion analysis of brane kinetic terms for gauge fields and the graviton in the Randall–Sundrum background, including the spectrum and the precision-electroweak consequences; with Carena et al. (2003) the reference for the price list of Open Problem O15. First cited in §17.

X. Dai and D. S. Freed. *Eta invariants and determinant lines*, *Journal of Mathematical Physics* **35** (1994) 5155–5194.

Relevance. Gives the modern determinant-line and eta-invariant framework in which global anomalies are properly formulated on mapping tori and their eight-dimensional fillings. In A3-BI6 the Atiyah–Patodi–Singer / Dai–Freed formalism is the technical backbone of §6: the global anomaly of BI-6 on the intended GG-6 background class is evaluated as the holonomy of an anomaly line on the seven-dimensional mapping torus and reduced to an eight-dimensional Dai–Freed integral that vanishes identically. First cited in §2.

J. Davighi, B. Gripaos, and N. Lohitsiri. *Global anomalies in the Standard Model(s) and beyond*, *JHEP* **07** (2020) 232, arXiv:1910.11277.

Relevance. Computes the spin-bordism groups that classify global anomalies for the Standard-Model gauge group and its extensions; for a direct-product group $SU(3) \times SU(2) \times U(1)^m$ the five-dimensional torsion is a single $\mathbb{Z}_2$, generated by Witten’s $SU(2)$ class. Cited in Subsection 11.6, where it closes the global-anomaly question for four-dimensional baryonic completions and locates the fork with BI-6 entirely in structure. First cited in §11.

M. Dierigl and M. Tartaglia. *(Quadratically) refined discrete anomaly cancellation*, *JHEP* **2508** (2025) 145, arXiv:2504.02934.

Relevance. Develops quadratically refined discrete Green–Schwarz mechanism in 6D F-theory: the most recent advance in global anomaly technology applicable to both SUSY and non-SUSY settings. In A3-BI6 we cite Dierigl–Tartaglia at the discussion of global-anomaly methods in §3, at the future-mathematics paragraph of §6, and in the open problem of §17 that asks for non-SUSY extension of the Monnier–Moore and Dierigl–Tartaglia technology to the unrestricted four-component BI-6 setting. First cited in §5.

M. Dine, N. Seiberg, and E. Witten. *Fayet–Iliopoulos terms in string theory*, *Nucl. Phys. B* **289** (1987) 589.

Relevance. The four-dimensional Green–Schwarz mechanism for an anomalous $U(1)$: the axion partner of the dilaton shifts under the gauge transformation and is eaten, leaving a massive vector and a Fayet–Iliopoulos term. Cited in §5.7 as the four-dimensional descendant that BI-6 declines on the flux computation $\Phi_B = 0$. First cited there.

L. J. Dixon and J. A. Harvey. *String theories in ten dimensions without spacetime supersymmetry*, *Nucl. Phys. B* **274** (1986) 93–105.

Relevance. The independent construction of the ten-dimensional non-supersymmetric heterotic strings, including the tachyon-free $O(16) \times O(16)$ theory, by twisting the supersymmetric heterotic string. Cited beside (Alvarez-Gaumé et al., 1986) at the supersymmetry contrast of §10: the two papers jointly established that the Green–Schwarz mechanism does not consume supersymmetry in 10D.

L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten. *Strings on orbifolds*, *Nucl. Phys. B* **261** (1985) 678.

Relevance. Introduces orbifold compactification in string theory: the systematic construction of consistent string vacua on orbifolds M/Γ with twisted sectors that restore modular invariance. In A3-BI6 we cite DHVW at the non-SUSY-heterotic paragraphs of §11; the interval and orbifold boundary conditions used in the 6D bulk $M_4 \times I_r \times S_\theta^1$ of GG-6 share the mathematical framework while replacing the string-theoretic origin with a constitutional one. First cited in §11.

C. Ehresmann. *Les connexions infinitésimales dans un espace fibré différentiable*, in *Colloque de Topologie, Bruxelles* (1950) 29.

Relevance. Founding definition of a connection on a fiber bundle, whose curvature obeys the Bianchi identity underlying Chern–Weil theory (Step 1 of §5.2). First cited in §5.

J. Erler. *Anomaly cancellation in six dimensions*, *J. Math. Phys.* **35** (1994) 1819, arXiv:hep-th/9304104.

Relevance. Systematically tabulates the gravitational, gauge, and mixed anomaly contributions of the six-dimensional $\mathcal{N} = (1, 0)$ multiplets — the per-field coefficients behind the $n_H - n_V + 29 n_T = 273$ constraint. In A3-BI6 we cite Erler at the gravitino-closure discussion of §7.5, where the cohort’s non-supersymmetric completion is contrasted with the tabulated SUGRA budget. First cited in §7.5.

D. S. Freed and M. J. Hopkins. *Reflection positivity and invertible topological phases*, *Geometry & Topology* **25** (2021) 1165–1330.

Relevance. Places anomaly cancellation in the modern reflection-positive invertible-field-theory language and clarifies the global/topological meaning of anomaly inflow via the cobordism classification of invertible TQFTs. It sits at the foundation of the global-anomaly methodology used in §6; listed here as background reading for readers approaching the differential-cohomology global completion of BI-6 from the reflection-positive-invertible side of the modern anomaly literature. First cited in the Executive Summary.

K. Fujikawa and H. Suzuki. *Path Integrals and Quantum Anomalies*, Oxford University Press (2004).

Relevance. The book-length path-integral treatment of anomalies: the non-invariance of the fermion measure as the origin of the index density, the viewpoint invoked in the physics-intuition opening of Appendix B. First cited in the Part VII course introduction.

I. García-Etxebarria and M. Montero. *Dai–Freed anomalies in particle physics*, *JHEP* **1908** (2019) 003, arXiv:1808.00009.

Relevance. Deploys the Dai–Freed formalism on the gauge groups of particle physics — the closest published analogue of the global checks run in Appendix I, and the recommended worked companion to its eta-invariant primer. First cited in the Part VII course introduction.

H. Georgi and S. Glashow. *Unity of all elementary-particle forces*, *Phys. Rev. Lett.* **32** (1974) 438.

Relevance. Introduces the SU(5) grand-unified theory: the unification of $SU(3)_c \times SU(2)_L \times U(1)_Y$ into a single semisimple gauge group with one matter generation in $\bar{\mathbf{5}} \oplus \mathbf{10}$. SU(5) is the non-abelian factor of the four-component bulk gauge group $\text{Spin}(1, 5) \times SU(5) \times U(1)_B \times U(1)_{Qi}$; the $\text{Tr } F_5^2$ contribution to the BI-6 four-form X_4 of §5.1 is its anomaly-polynomial signature. First cited in §5.

H. Georgi. *Lie Algebras in Particle Physics*, 2nd ed., Perseus Books (1999).

Relevance. The working physicist’s Lie-algebra text; supplies the derivations behind the representation indices (A, A_3, B, C) tabulated and used in Appendix F. First cited in the Part VII course introduction.

V. Gorini, A. Kossakowski, and E. C. G. Sudarshan. *Completely positive dynamical semigroups of N -level systems*, *J. Math. Phys.* **17** (1976) 821.

Relevance. Derives the GKLS generator for completely positive Markovian quantum dynamical semigroups on N -level systems — one half of the canonical open-systems classification. In A3-BI6 we cite Gorini–Kossakowski–Sudarshan alongside Lindblad at §6.6, where the Markovian limit of the boundary OGGEF channel of GG-A5-GDOS is identified as exactly the GKLS structure; the bulk BI-6 anomaly freedom proved in A3-BI6 is the lawfulness prerequisite for that boundary channel. First cited in §6.

M. B. Green and J. H. Schwarz. *Anomaly cancellations in supersymmetric $D = 10$ gauge theory and superstring theory*, *Phys. Lett. B* **149** (1984) 117.

Relevance. Discovers the Green–Schwarz mechanism in ten dimensions: anomaly cancellation through a higher-form tensor whose modified Bianchi identity $dH_3 = X_4$ couples to the gauge curvature, with $I_{12} = X_4 \wedge X_8$ factorization. The six-dimensional BI-6 construction of A3-BI6 is the axiomatic descendant of this 1984 result: §10 carries out the feature-by-feature comparison, and §10.7 reads BI-6 as the GGEF completion of GS within the GG-Theory Constitution. First cited in the Executive Summary.

M. B. Green, J. H. Schwarz, and P. C. West. *Anomaly-free chiral theories in six dimensions*, *Nucl. Phys. B* **254** (1985) 327.

Relevance. First construction of anomaly-free chiral gauge theories in six dimensions using the Green–Schwarz mechanism with the factorization $I_8 = (1/2)X_4 \wedge X_4$. The direct precursor of every 6D anomaly construction including BI-6. In A3-BI6 we cite GSW at the 6D specialization paragraphs of §3, at the existing-literature comparison in §11, and at the rank-one and Diophantine matching discussions of §7 where the perfect-square factorization is rederived for the four-component coefficient vector. First cited in the Executive Summary.

M. B. Green, J. H. Schwarz, and E. Witten. *Superstring Theory, Vol. II: Loop Amplitudes, Anomalies and Phenomenology*, Cambridge University Press (1987).

Relevance. The original textbook treatment of the ten-dimensional Green–Schwarz mechanism (Ch. 13), including the $SO(n)$ trace identities and the explicit X_8 ; the primary reference for Appendix C, whose derivations the student should check against it. First cited in the Part VII course introduction.

S. S. Gubser, I. R. Klebanov, and A. M. Polyakov. *Gauge theory correlators from non-critical string theory*, *Phys. Lett. B* **428** (1998) 105, arXiv:hep-th/9802109.

Relevance. Makes the AdS/CFT correspondence computationally precise via the GKP dictionary that relates bulk fields to boundary operators by matching the on-shell bulk action to the generating functional of boundary correlators. In A3-BI6 we cite Gubser–Klebanov–Polyakov in §12 alongside Maldacena and Witten 1998; the dictionary underlies the OGGEF operational map that the boundary GDOS program implements as a CPTP channel, with BI-6’s anomaly freedom providing the bulk consistency that the dictionary presupposes. First cited in §12.

Y. Hamada and G. J. Loges. *Enumerating 6D supergravities with $T \leq 1$* , *JHEP* **2412** (2024) 167, arXiv:2404.08845.

Relevance. Systematic enumeration of all anomaly-free 6D $\mathcal{N} = (1, 0)$ supergravity models with $T \leq 1$ tensor multiplets: the state-of-the-art census of the 6D SUSY landscape. In A3-BI6 we cite Hamada–Loges at the landscape-enumeration paragraphs of §11 and at the open-problem on extending the non-SUSY classification in §17, where adapting their methods to the unrestricted four-component BI-6 setting is identified as a well-posed mathematical problem. First cited in §11.

S. de Haro, S. N. Solodukhin, and K. Skenderis. *Holographic renormalization*, *Commun. Math. Phys.* **217** (2001) 595, arXiv:hep-th/0002230.

Relevance. Develops holographic renormalization: the systematic procedure for regulating the on-shell bulk action and extracting boundary correlators, with the radial coordinate identified as RG depth. In A3-BI6 we cite de Haro–Skenderis–Solodukhin in the AdS/CFT review of §12; the radial-RG identification they made precise is adopted constitutionally in GG-6 via postulate POS-5 (radial RG depth) and reappears in the holographic-comparison table of §12. First cited in §12.

J. A. Harvey. *TASI 2003 lectures on anomalies*, arXiv:hep-th/0509097.

Relevance. The standard TASI lectures on anomalies and inflow, the second recommended lecture-note companion (beside Bilal, 2008) to Appendices B and C. First cited in the Part VII course introduction.

H. Hayashi, H.-C. Kim, and M. Kim. *Spectra of BPS strings in 6D supergravity and the swampland*, *JHEP* **2503** (2025) 123, arXiv:2310.12219.

Relevance. Tests Weak-Gravity and Emergent-String swampland conjectures using BPS-string spectra in 6D $\mathcal{N} = (1, 0)$ supergravity — the most recent systematic swampland analysis in 6D. In A3-BI6 we cite Hayashi–Kim–Kim at the swampland paragraphs of §11; the comparison is asymmetric — BI-6 achieves consistency without invoking SUSY multiplets or BPS strings, by relying instead on the four-component coefficient lock and the OGN-5 integer ledger derived in §9. First cited in §11.

M. J. Hopkins and I. M. Singer. *Quadratic functions in geometry, topology, and M-theory*, *J. Differential Geom.* **70** (2005) 329, arXiv:math/0211216.

Relevance. Develops the differential-cohomology framework for quadratic refinements of self-dual abelian gauge fields, originally aimed at the M-theory C_3 -field: the mathematical machinery that makes the self-dual tensor B_2 of BI-6 single-valued on non-trivial topology. In A3-BI6 we cite Hopkins–Singer at the global-completion paragraphs of §6 and at the quadratic-refinement discussion; the Wu–Chern–Simons / differential-cohomology refinement of the integral X_4 class derived there is the direct application of their construction. First cited in the Executive Summary.

L. E. Ibáñez, R. Rabadán, and A. M. Uranga. *Anomalous $U(1)$ ’s in type I and type IIB $D = 4$, $N = 1$ string vacua*, *Nucl. Phys. B* **542** (1999) 112 [hep-th/9808139].

Relevance. The generalized four-dimensional Green–Schwarz mechanism of open-string constructions, in which several anomalous $U(1)$ s are canceled by twisted axions that are eaten. Cited with Dine, Seiberg, and Witten, 1987 in §5.7. First cited there.

L. E. Ibáñez and A. M. Uranga. *String Theory and Particle Physics: An Introduction to String Phenomenology*, Cambridge University Press (2012).

Relevance. The phenomenological bridge between string constructions and particle physics; recommended to the student comparing the string-side model-building strategies with the constitutional route of this paper (Part IV and Appendix C). First cited in the Part VII course introduction.

J. Kaidi, Y. Tachikawa, and K. Yonekura. *On non-supersymmetric heterotic branes*, *JHEP* **2503** (2025) 211, arXiv:2411.04344.

Relevance. Studies non-supersymmetric heterotic branes, demonstrating that string theory accommodates non-supersymmetric configurations of direct relevance to the GG-6 philosophy of doing 6D physics without 6D supersymmetry. In A3-BI6 we cite Kaidi–Tachikawa–Yonekura at the non-SUSY-heterotic paragraphs of §11; the convergence point is that anomaly cancellation continues to be the principal admissibility filter once supersymmetry is removed, exactly as in the four-component BI-6 construction of §5. First cited in §11.

H.-C. Kim, C. Vafa, and K. Xu. *Finite landscape of 6D $\mathcal{N} = (1,0)$ supergravity*, arXiv:2411.19155.

Relevance. Argues for finiteness of the complete 6D $\mathcal{N} = (1,0)$ supergravity landscape — the latest evidence that the 6D string landscape is bounded. In A3-BI6 we cite Kim–Vafa–Xu at the finite-versus-infinite paragraphs of §10 and at the swampland and landscape paragraphs of §11; the OGN-5 integer ledger of §9 delivers an even sharper finiteness statement from inside the GG-6 framework, namely a single Diophantine solution rather than a bounded set. First cited in §11.

V. Kumar, D. R. Morrison, and W. Taylor. *Global aspects of the space of 6D $\mathcal{N} = 1$ supergravities*, *JHEP* **1011** (2010) 118, arXiv:1008.1062.

Relevance. Systematic classification of the global structure of anomaly-free 6D $\mathcal{N} = (1,0)$ supergravity theories, including the integral anomaly lattice underpinning the string universality program. In A3-BI6 we cite Kumar–Morrison–Taylor at the F-theory and lattice paragraphs of §11.2 and at the comparison in §11; the contrast with the OGN-5 integer ledger $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ derived in §9 is the central technical point of the comparison. First cited in the Executive Summary.

V. Kumar, D. S. Park, and W. Taylor. *6D supergravity without tensor multiplets*, *JHEP* **1104** (2011) 080, arXiv:1011.0726.

Relevance. Studies 6D supergravity theories without tensor multiplets ($T = 0$), demonstrating the additional constraints from anomaly cancellation on the simplest class. In A3-BI6 we cite Kumar–Park–Taylor at the F-theory paragraphs of §11.2; the contrast is structural — BI-6 carries exactly one self-dual tensor B_2 enforced by axiom AX-4, neither none nor many, with the corresponding rank-one anomaly matrix derived in §5 and the perfect-square factorization $I_8 = (1/2)X_4^2$ following uniquely. First cited in §11.

V. Kumar and W. Taylor. *String universality in six dimensions*, *Adv. Theor. Math. Phys.* **15** (2011) 325, arXiv:0906.0987.

Relevance. Proposes the string universality conjecture: every anomaly-free 6D $\mathcal{N} = (1, 0)$ supergravity theory with one tensor multiplet is realized somewhere in string theory. The sharpest statement of the string program’s completeness claim in 6D. In A3-BI6 we cite Kumar–Taylor at the discussion of finite versus infinite landscape in §10 and §11; the OGN-5 Diophantine spectrum problem of §8 is the constitutional analogue of universality from inside the GG-6 framework. First cited in §11.

G. Lindblad. *On the generators of quantum dynamical semigroups*, *Commun. Math. Phys.* **48** (1976) 119.

Relevance. Derives the most general bounded generator of a quantum dynamical semigroup: the other half (alongside Gorini–Kossakowski–Sudarshan) of the canonical GKLS classification of completely positive Markovian open-system dynamics. In A3-BI6 we cite Lindblad at §6.6, where the Markovian limit of the boundary OGGE channel of GG-A5-GDOS is identified as exactly the GKLS–Lindblad structure that the bulk anomaly-freedom certificate of BI-6 licenses. First cited in §6.

G. J. Loges. *New infinite class of 6D $\mathcal{N} = (1, 0)$ supergravities*, *Phys. Rev. D* **109** (2024) 126006, arXiv:2402.04371.

Relevance. Finds new infinite families of anomaly-free 6D $\mathcal{N} = (1, 0)$ supergravity models, demonstrating that the SUSY landscape is still actively expanding. In A3-BI6 we cite Loges at the 6D-landscape paragraphs of §11 and at the discussion of finiteness in §17; the contrast emphasizes why the bounded Diophantine spectrum derived from the OGN-5 integer ledger in §9 represents a categorically different approach from infinite-family enumeration within SUSY. First cited in §11.

J. Maldacena. *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231, arXiv:hep-th/9711200.

Relevance. Proposes the AdS/CFT correspondence: the foundational result that a d -dimensional CFT is dual to a $(d + 1)$ -dimensional bulk gravity theory. The holographic logic that a radial bulk coordinate encodes boundary RG scale is the direct ancestor of the GG-6 radial coordinate $r = \ln(E_{\text{GUT}}/\mu)$. In A3-BI6 we cite Maldacena throughout §12, especially in the comparison table that locates AdS/CFT, Randall–Sundrum, and BI-6 all on Stratum 2 of the GG-6 constitutional classification. First cited in §12.

J. C. Maxwell. *A dynamical theory of the electromagnetic field*, *Phil. Trans. Roy. Soc. London* **155** (1865) 459–512.

Relevance. Originates the gauge-invariant four-vector potential of classical electrodynamics: the historical seed of the gauge invariance whose chiral-fermion failure at one loop is the Adler–Bell–Jackiw anomaly that BI-6 must cancel in higher dimensions. In A3-BI6 we cite Maxwell at the historical introduction of §1 and in the gauge-theory genealogy of §3; the modified Bianchi identity $dH_3 = X_4$ of BI-6 is the topological completion of the Maxwell–Yang–Mills tradition. First cited in §3.

J. McNamara and C. Vafa. *Cobordism classes and the swampland*, arXiv:1909.10355.

Relevance. The origin of the cobordism conjecture: quantum-gravity consistency requires cobordism classes to vanish, with defects supplied where they otherwise would not. Cited at the swampland audit of §13 and in the cobordism-defect objection of Part IX, where the cohort’s obligation — exhibit the defects or show the classes vanish — is registered against it. First cited in §13.

J. Milnor and J. Stasheff. *Characteristic Classes*, Princeton University Press (1974).

Relevance. Standard textbook source for Pontryagin classes, characteristic elements, and the trace-to-characteristic-class conversions used throughout the BI-6 proof. Milnor–Stasheff is the canonical reference for the characteristic-class identities deployed in Appendix B when converting $\text{tr } R^2$, $\text{Tr } F_5^2$, F_B^2 , and $F_{Q_i}^2$ into the four-component four-form X_4 whose perfect-square factorization $I_8 = (1/2)X_4^2$ closes the local anomaly of BI-6. First cited in §5.

M. Mimura and H. Toda. *Homotopy groups of SU(3), SU(4) and Sp(2)*, *Journal of Mathematics of Kyoto University* **3** (1963) 217–250.

Relevance. Supplies $\pi_6(\text{SU}(3)) = \mathbb{Z}_6$ and $\pi_6(\text{SU}(4)) = 0$; with Toda’s $\pi_6(\text{SU}(2)) = \pi_6(S^3) = \mathbb{Z}_{12}$ (Toda, 1962) these are the unstable groups that any six-dimensional lift of a Standard-Model-subgroup gauge content must confront — the obstruction the stable-range fact $\pi_6(\text{SU}(5)) = 0$ spares BI-6. Cited at the hypothesis-(H2) contrast in Subsection 11.6 and in the remark following Theorem 6.13. First cited in §11.

S. Monnier. *The anomaly field theories of six-dimensional (2, 0) superconformal theories*, *Adv. Theor. Math. Phys.* **22** (2018) 2035–2089, arXiv:1706.01903.

Relevance. Develops the anomaly-field-theory viewpoint for six-dimensional chiral systems: a close conceptual parallel for the tensor-sector analysis that A3-BI6 carries out for BI-6. It underlies the modern invertible-field-theory framework used in §6 to evaluate the global anomaly on mapping tori; it is listed here as background reading for readers approaching the differential-cohomology completion of BI-6 from the (2, 0)-superconformal anomaly literature. First cited in §2.

S. Monnier and G. W. Moore. *Remarks on the Green–Schwarz terms of six-dimensional supergravity theories*, *Commun. Math. Phys.* **372** (2019) 963, arXiv:1808.01334.

Relevance. Constructs the Green–Schwarz counterterms on non-trivially topological spacetimes and proves global anomaly cancellation for standard 6D $\mathcal{N} = (1, 0)$ supergravity gauge groups. The definitive treatment of global consistency that A3-BI6 must satisfy on the intended GG-6 background class. We cite Monnier–Moore at the global-anomaly paragraphs of §6, at the quadratic-refinement discussion, and at the future-mathematics open problem on non-SUSY extension in §17. First cited in the Executive Summary.

D. R. Morrison and W. Taylor. *Classifying bases for 6D F-theory models*, *Central Eur. J. Phys.* **10** (2012) 1072, arXiv:1201.1943.

Relevance. Classifies the base surfaces for F-theory compactifications to 6D: a key step in the enumeration of the 6D F-theory landscape. In A3-BI6 we cite Morrison–Taylor 2012a at the F-theory enumeration paragraphs of §11.2; the contrast underscores how F-theory builds 6D

models bottom-up from base-geometry data, while BI-6 builds them top-down from constitutional axioms and lands on the unique four-component coefficient vector derived in §9. First cited in §11.

D. R. Morrison and W. Taylor. *Matter and singularities*, *JHEP* **1201** (2012) 022, arXiv:1106.3563.

Relevance. Develops the correspondence between singularity types in elliptic fibrations and matter representations in F-theory: the geometric mechanism for determining 6D matter spectra. In A3-BI6 we cite Morrison–Taylor 2012b alongside Morrison–Taylor 2012a at the F-theory paragraphs of §11.2; the comparison contrasts the F-theory geometry-to-spectrum pathway with the constitutional axiom-to-spectrum pathway through which BI-6 determines the $SU(5)$ matter content of §9. First cited in §11.

D. R. Morrison and C. Vafa. *Compactifications of F-theory on Calabi–Yau threefolds I*, *Nucl. Phys. B* **473** (1996) 74, arXiv:hep-th/9602114.

Relevance. Develops F-theory compactification on elliptic Calabi–Yau threefolds to six dimensions, providing the geometric origin of 6D gauge groups and matter from fiber singularities. In A3-BI6 we cite Morrison–Vafa I at the F-theory comparison of §11.2; the contrast is that BI-6 arrives at the four-component bulk gauge group $\text{Spin}(1, 5) \times SU(5) \times U(1)_B \times U(1)_{Q_i}$ constitutionally rather than from CY-fibration data, with the trace-ratio lock $|a/c| = |a/d| = 5/2$ derived in §9. First cited in §11.

D. R. Morrison and C. Vafa. *Compactifications of F-theory on Calabi–Yau threefolds II*, *Nucl. Phys. B* **476** (1996) 437, arXiv:hep-th/9603161.

Relevance. Completes the F-theory compactification program to six dimensions and establishes the anomaly-lattice correspondence used throughout the string-universality program. In A3-BI6 we cite Morrison–Vafa II alongside Part I at the F-theory paragraphs of §11.2; the lattice correspondence on the F-theory side parallels the OGN-5 integer ledger derived in §9 but achieves bounded multiplicity rather than the unique Diophantine solution that BI-6 forces under the constitutional admissibility filter. First cited in §11.

M. K. Murray. *Bundle gerbes*, *J. London Math. Soc.* **54** (1996) 403.

Relevance. The rigorous object carrying a two-form potential B_2 with three-form curvature H_3 and Dixmier–Douady class in H^3 ; the precise “one level up” bundle of Step 2 of §5.2. First cited in §5.

M. Nakahara. *Geometry, Topology and Physics*, 2nd ed., Institute of Physics Publishing (2003).

Relevance. The standard physicist’s text for characteristic classes, index theorems, and the eta invariant (Ch. 11 and 12) — the geometry background assumed by Appendices H and I. First cited in the Part VII course introduction.

S. Nakajima. *On quantum theory of transport phenomena*, *Prog. Theor. Phys.* **20** (1958) 948.

Relevance. Introduces the projection-operator technique for deriving reduced (non-Markovian) dynamics from a closed-system Liouville evolution: the mathematical foundation of the OGGEF channel used in the boundary GDOS program. Nakajima’s 1958 result is one of the foundational

components of the open-system formalism that A3-BI6 makes self-consistent through the bulk anomaly-freedom proof; listed here as background reading for the OGGEF discussion in §1.5 and §6.6. First cited in §1.

H. Nishino and E. Sezgin. *Matter and gauge couplings of $\mathcal{N} = 2$ supergravity in six dimensions*, *Phys. Lett. B* **144** (1984) 187.

Relevance. Foundational construction of 6D $\mathcal{N} = (1, 0)$ matter-coupled supergravity: the starting point for all subsequent 6D SUGRA anomaly analyzes. In A3-BI6 we cite Nishino–Sezgin at the 6D SUGRA paragraphs of §11; the comparison there shows that BI-6 achieves anomaly freedom without invoking 6D supersymmetry by replacing the SUGRA multiplet structure with the OGN-5 integer ledger and the four-component coefficient vector pinned by the trace-ratio theorem of §9. First cited in §11.

M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory*, Addison-Wesley (1995).

Relevance. The standard first-year graduate QFT text; its Ch. 19 treatment of the axial anomaly is the assumed entry point for the Part VII course. First cited in the Part VII course introduction.

J. Polchinski. *String Theory, Vol. I: An Introduction to the Bosonic String*, Cambridge University Press (1998).

Relevance. Standard graduate-level reference for the bosonic string, the worldsheet formalism, and the conformal-field-theory machinery the anomaly discussion presupposes. The Green–Schwarz mechanism itself is in Volume II. Polchinski Vol. I is the canonical pedagogical source for readers approaching BI-6 from the string-theory tradition; the comparison of §10 relies on the standard 10D treatments collected there, while BI-6 borrows the cohomological machinery and discards the stringy ontology. First cited in §12.

L. Randall and R. Sundrum. *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370.

Relevance. Introduces warped extra dimensions: a 5D bulk with exponential warp factor that generates the electroweak/Planck hierarchy. The GG-6 throat geometry of $M_4 \times I_r \times S_\theta^1$ uses the same warp mechanism, with the warp factor responsible for the OGN-5 hierarchy $M_{\text{lock}} \sim M_{\text{Pl}} e^{-K}$. In A3-BI6 we cite Randall–Sundrum at the warped-geometry paragraphs of §12, where the constitutional reading places Randall–Sundrum on Stratum 2 alongside BI-6. First cited in §12.

V. Sadov. *Generalized Green–Schwarz mechanism in F-theory*, *Phys. Lett. B* **388** (1996) 45, arXiv:hep-th/9606008.

Relevance. Extends the Green–Schwarz mechanism to the F-theory context with multiple tensor multiplets, demonstrating the generalization that BI-6 explicitly avoids by carrying exactly one self-dual tensor B_2 . Listed in A3-BI6 as background for the F-theory comparison of §11.2 and Objection 5 of §IX; the rank-one corollary of Theorem 5.1 confirms that the microscopic BI-6 anomaly matrix collapses to rank one, so multi-tensor extensions in the spirit of Sadov are unforced and inadmissible under AX-4. First cited in §7.

A. Sagnotti. *A note on the Green–Schwarz mechanism in open-string theories*, *Phys. Lett. B* **294** (1992) 196.

Relevance. Extends the Green–Schwarz mechanism to open-string and 6D settings with anti-self-dual tensor multiplets, and is widely cited for multi-tensor constructions $I_8 = (1/2) \sum_i X_4^{(i)} \wedge X_4^{(i)}$. In A3-BI6 we cite Sagnotti at the 6D-literature paragraphs of §11 and at Objection 5 of §IX, where multi-tensor extensions are shown to be excluded by axiom AX-4 (MSMF) and by the rank-one theorem of §5. First cited in the Executive Summary.

J. H. Schwarz. *Anomaly-free supersymmetric models in six dimensions*, *Phys. Lett. B* **371** (1996) 223–230, arXiv:hep-th/9512053.

Relevance. Compact statement of the six-dimensional $\mathcal{N} = (1, 0)$ anomaly-cancellation conditions and the multiplet bookkeeping behind the 273 constraint. In A3-BI6 we cite Schwarz alongside Erler at §7.5, where the cohort’s anti-self-dual-tensor/singlet-fermion closure is compared with the supersymmetric budget it replaces. First cited in §7.5.

N. Seiberg and W. Taylor. *Charge lattices and consistency of 6D supergravity*, *JHEP* **1106** (2011) 001, arXiv:1103.0019.

Relevance. Establishes the charge-lattice consistency conditions for 6D $\mathcal{N} = (1, 0)$ supergravity: the integrality requirements that fix admissible matter content. These are the SUSY analogue of the OGN-5 integer-lattice constraints in GG-6. In A3-BI6 we cite Seiberg–Taylor at the global-refinement paragraphs of Appendix I, where the differential-cohomology refinement of the integral X_4 class is discussed; the rank-one structure of the BI-6 tensor lattice makes the charge-quantization analysis significantly more constrained than in the SUSY setting. First cited in the Executive Summary.

R. Slansky. *Group theory for unified model building*, *Phys. Rept.* **79** (1981) 1.

Relevance. The canonical tables of group-theory data — dimensions, indices, branching rules — from which the SU(5) representation data of Appendix F can be independently checked. First cited in the Part VII course introduction.

R. Stora. *Algebraic structure and topological origin of anomalies*, in *Progress in Gauge Field Theory* (Cargèse 1983), NATO ASI Series B **115**, Plenum (1984) 543–562.

Relevance. With Zumino, the origin of the descent formalism — the chain of forms relating an anomaly polynomial in $2n + 2$ dimensions to the anomaly in $2n$ dimensions — which is the machinery of Appendix B and of every local cancellation argument in this paper. Named in the text as the Stora–Zumino descent. First cited in Section 3.

R. E. Stong. *Notes on Cobordism Theory*, Princeton University Press (1968).

Relevance. Provides the classical cobordism facts used in the automatic-admissibility theorem, in particular the structure of the low-dimensional spin bordism groups. String bordism post-dates this volume by two decades and is not taken from it; the string-bordism and tmf input is cited separately. Stong’s textbook is the standard source for the bordism-theoretic ingredients used in the determinant-line and eta-invariant computations of Appendix I; listed here as background

reading for readers approaching the global completion of BI-6 from the bordism side. First cited in §5.

H. Toda. *Composition Methods in Homotopy Groups of Spheres*, Annals of Mathematics Studies 49, Princeton University Press (1962).

Relevance. The source of $\pi_6(S^3) = \mathbb{Z}_{12}$, hence $\pi_6(\mathrm{SU}(2)) = \mathbb{Z}_{12}$, the unstable group a six-dimensional lift of the Standard-Model group meets and $\mathrm{SU}(5)$ does not. Cited beside Mimura and Toda, 1963 at every $\pi_6(\mathrm{SU}(2))$ site. First cited in §11.

H.-C. Tarazi and C. Vafa. *On the finiteness of 6D supergravity landscape*, arXiv:2106.10839.

Relevance. Proves finiteness bounds on the 6D supergravity landscape: the string-theory analogue of the bounded Diophantine spectrum problem in GG-6. In A3-BI6 we cite Tarazi–Vafa at the landscape-finiteness paragraphs of §11 and at the swampland discussion; the parallel is structural — both programs constrain the admissible 6D theory space, but the OGN-5 ledger of §9 closes the count to a unique solution $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ rather than to a bounded set. First cited in §11.

W. Taylor. *TASI lectures on supergravity and string vacua in various dimensions*, arXiv:1104.2051.

Relevance. Comprehensive TASI-lectures review of 6D supergravity, anomaly cancellation, and the F-theory connection: an essential reference for any 6D anomaly-cancellation paper. In A3-BI6 we cite Taylor at the F-theory paragraphs of §11.2, the existing-literature review of §11, and the global-refinement discussion of Appendix I; readers unfamiliar with the 6D SUSY-anomaly tradition will find Taylor’s lectures the natural entry point before working through the four-component BI-6 construction. First cited in §7.

W. Taylor and Y.-N. Wang. *The F-theory geometry with most flux vacua*, *JHEP* **12** (2015) 164, arXiv:1511.03209.

Relevance. The origin of the $10^{272\,000}$ flux-vacua count, for a single elliptically fibered F-theory geometry. Cited at the landscape-count paragraphs of §11, where the attribution ladder of vacuum estimates is stated. First cited in §11.

C. Vafa. *Evidence for F-theory*, *Nucl. Phys. B* **469** (1996) 403, arXiv:hep-th/9602022.

Relevance. Introduces F-theory as a non-perturbative framework for type IIB string compactifications: the geometric foundation for systematic 6D anomaly-free model building from elliptically fibered Calabi–Yau threefolds. In A3-BI6 we cite Vafa at the F-theory paragraphs of §11.2; the comparison emphasizes that F-theory determines 6D gauge groups and matter from fiber-singularity geometry, whereas BI-6 determines them constitutionally through axioms AX-1–AX-4 and the OGN-5 ledger of §9. First cited in §11.

C. Vafa. *The string landscape and the swampland*, arXiv:hep-th/0509212.

Relevance. Introduces the swampland concept: the distinction between consistent-looking effective theories and those actually realized in string theory. In A3-BI6 we cite Vafa 2005 alongside Brennan–Carta–Vafa at the swampland paragraphs of §11 and at the landscape discussion of §10; the swampland framework provides the conceptual contrast against which the BI-6 construction

is positioned — not as a swampland counter-example but as a constitutionally selected single point. First cited in §11.

S. Weinberg. *The Quantum Theory of Fields, Vol. II: Modern Applications*, Cambridge University Press (1996).

Relevance. Chapter 22 is the deepest textbook treatment of anomalies in print — consistent versus covariant forms, anomaly matching, and the descent equations; the recommended in-depth companion to Appendix B. First cited in the Part VII course introduction.

J. Wess and B. Zumino. *Consequences of anomalous Ward identities*, *Phys. Lett. B* **37** (1971) 95.

Relevance. Establishes the Wess–Zumino consistency condition: the algebraic identity that any genuine anomaly must satisfy and that makes the descent formalism well-defined. In A3-BI6 the descent chain from the eight-form anomaly polynomial I_8 to the six-form variation, used to verify the Green–Schwarz cancellation in §5 and worked out explicitly in Appendix B, is the systematic implementation of this consistency condition for the four-component BI-6 four-form X_4 . First cited in the Executive Summary.

E. Witten. *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253, arXiv:hep-th/9802150.

Relevance. Establishes the on-shell bulk gravitational action as the generating functional of boundary correlators in the AdS/CFT dictionary: the functional-integral formulation that GG-6/GDOS reads operationally as a CPTP channel rather than as an analytic-continuation identity. In A3-BI6 we cite Witten 1998 alongside Maldacena and Gubser–Klebanov–Polyakov in the AdS/CFT review of §12; the constitutional reading of §12 places AdS/CFT and BI-6 together on Stratum 2 of the GG-Theory Constitution. First cited in §12.

E. Witten. *Fermion path integrals and topological phases*, *Rev. Mod. Phys.* **88** (2016) 035001, arXiv:1508.04715.

Relevance. The modern physicist’s treatment of the eta invariant, the APS theorem, and anomaly inflow — the recommended companion to the eta primer of Appendix I, including the twisted circle example reproduced there. First cited in the Part VII course introduction.

T. T. Wu and C. N. Yang. *Concept of nonintegrable phase factors and global formulation of gauge fields*, *Phys. Rev. D* **12** (1975) 3845.

Relevance. The fiber-bundle dictionary for gauge theory: holonomy as the Wilson line, and the Dirac monopole charge as a bundle Chern number — the integrality invoked in Step 6 of §5.2. First cited in §5.

R. Zwanzig. *Ensemble method in the theory of irreversibility*, *J. Chem. Phys.* **33** (1960) 1338.

Relevance. Develops the memory-kernel equation for projected open-system dynamics, completing the Nakajima–Zwanzig framework. Zwanzig’s 1960 construction together with Nakajima 1958 yields the exact open-system evolution that the boundary GDOS program of GG-A5-GDOS implements as a CPTP channel; listed here as background reading for the OGGEF discussion in §1.5, whose lawfulness rests on the bulk BI-6 anomaly proof of A3-BI6. First cited in §1.

Douglas, M. R. (2003). The statistics of string/M theory vacua. *Journal of High Energy Physics*, **05**, 046.

Relevance. Develops the statistics of string/M-theory vacua: the quantitative side of the landscape program that estimates the number of consistent compactifications. In A3-BI6 we cite Douglas alongside Susskind in the inequivalent-compactifications paragraphs of §18; the contrast is methodological — the OGN-5 Diophantine spectrum of §9 closes the count to a unique solution rather than asymptoting to a statistical ensemble, replacing measure-theoretic landscape arguments with constitutional integer arithmetic. First cited in §12.

Einstein, A. (1915). Die Feldgleichungen der Gravitation. *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, 844–847.

Relevance. Establishes general relativity as the dynamical theory of the Lorentzian metric whose Riemann curvature obeys the Bianchi identity that BI-6 modifies. In A3-BI6 we cite Einstein 1915 in the classical-foundations enumeration of §18 and in the historical introduction; the geometric stratum on which the four-form $X_4 = \text{tr } R^2 - \text{Tr } F_5^2 - (2/5)F_B^2 - (2/5)F_{Q_i}^2$ is built combines Einstein’s Riemann-curvature contribution with the Yang–Mills gauge contributions in a single perfect-square factorization. First cited in §12.

Polchinski, J. (1998b). *String Theory, Vol. II: Superstring Theory and Beyond*. Cambridge University Press.

Relevance. Standard graduate-level reference for type II superstrings, heterotic strings, dualities, and the modern formulation of the 10D Green–Schwarz mechanism. Polchinski Vol. II is the canonical pedagogical source for the 10D constructions that A3-BI6 compares against in §10; listed here as background reading for the feature-by-feature comparison between the 10D heterotic GS factorization $I_{12} = X_4 \wedge X_8$ and the rank-one 6D BI-6 factorization $I_8 = (1/2)X_4^2$. First cited in §12.

Riemann, B. (1854). Über die Hypothesen, welche der Geometrie zu Grunde liegen. Habilitationsschrift, Göttingen. (Published posthumously: *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, **13**, 133–150, 1868.)

Relevance. Establishes the foundations of n -dimensional differential geometry in the Habilitationsschrift on the hypotheses underlying geometry: the mathematical setting that Einstein later identified as the physical arena of general relativity and that GG-Theory extends to six dimensions. In A3-BI6 we cite Riemann 1854 at the historical introduction and at the classical-foundations enumeration of §18, where the lineage runs from Riemann through Einstein, Bianchi, Yang–Mills, and Green–Schwarz to the modified Bianchi identity of BI-6. First cited in §2.

Susskind, L. (2003). The anthropic landscape of string theory. arXiv:hep-th/0302219.

Relevance. Argues for the anthropic landscape of string theory: the philosophical pivot from uniqueness to multiplicity that motivates much of contemporary string phenomenology. In A3-BI6 we cite Susskind alongside Douglas in the inequivalent-compactifications paragraphs of §18; the contrast is constitutional — the OGN-5 integer ledger derived in §9 selects a unique solution $(M, N, L; J, K) = (1, 3, 5; 7, 35)$, removing the anthropic multiplicity that Susskind’s landscape argument embraces. First cited in §12.

Yang, C. N., and Mills, R. L. (1954). Conservation of isotopic spin and isotopic gauge invariance. *Physical Review*, **96**, 191–195.

Relevance. Generalizes Maxwell’s gauge principle from abelian $U(1)$ to non-abelian $SU(2)$ via the connection one-form valued in a Lie algebra: the construction underpinning every modern gauge theory. In A3-BI6 we cite Yang–Mills in the classical-foundations enumeration of §18; the $SU(5)$ factor of the four-component BI-6 bulk gauge group is the direct non-abelian descendant, and the rank-one $\text{Tr } F_5^2$ contribution to the four-form X_4 derived in §5 is its anomaly-polynomial signature. First cited in the Executive Summary.

Almheiri, A., Hartman, T., Maldacena, J., Shaghoulian, E., & Tajdini, A. (2020). *Replica wormholes and the entropy of Hawking radiation*. *Journal of High Energy Physics*, **2020**, 13.

Relevance. The 2020 islands formula and replica-wormhole construction compute the Page curve of Hawking radiation as a saddle-point bulk gravitational integral, with the boundary entanglement entropy descending from a finite-area “island” region of the bulk. We cite it in Section 12.9 as the canonical example of a quantum-gravitational result in string theory that is not in the GGEP-closed sector and that the cohort framework does not subsume. First cited in §12.

Penington, G., Shenker, S. H., Stanford, D., & Yang, Z. (2020). *Replica wormholes and the black hole interior*. *JHEP* **03** (2022) 205, arXiv:1911.11977.

Relevance. Penington *et al.* 2020 (independently of and contemporaneously with Almheiri *et al.* (2020)) derived the replica-wormhole contribution to the gravitational path integral that produces the Page-curve transition. We cite it in Section 12.9 as part of the same body of quantum-gravitational string-theoretic results that the cohort framework explicitly does not subsume. First cited in §12.