UCLA/Arisaka-GG-P4-Real
v20.2 — August 22, 2026 — *Foundations Branch*

The Real Geometry Beneath the Complex Numbers of Physics

Positive Frequency is Holomorphy — from Cardano's Cubics to Twistor Space, via the Sixth Dimension of GG-Theory

Katsushi Arisaka

Department of Physics and Astronomy, Electrical and Computer Engineering,
University of California, Los Angeles, Los Angeles, CA 90095, USA

Contact: arisaka@physics.ucla.edu

Abstract

Complex numbers were invented in the sixteenth century to pass through the impossible square roots of the cubic; by 1925 they had become the working substance of physics, and the puzzle of why has stood ever since — posed at full strength by Penrose, for quantum amplitudes and for the holomorphic geometry of twistor theory alike. This preprint assembles a calibrated answer. Physics contains two complex structures, both derived from globally shared *real* data. The gauge phase of charged matter is a real angle on the compact, non-spatial phase circle of the GG-6 bulk: charge quantization is its compactness, the Aharonov–Bohm phase its holonomy, the imaginary unit its rotation generator. The state-space i is the polar phase of the constitutional clock — the Hilbert transform, a real operator — selected uniquely by causality and energy positivity; the two coincide on charged sectors exactly under the Feynman–Stückelberg pairing, so antiparticles are the consistency condition between circle and clock. The same selection extends, conditionally, to interacting matter — and to holomorphy itself: positive frequency is Hardy-space holomorphy, forward-tube analyticity its field form, and the twistor division $\mathbb{PT}^+/\mathbb{PT}^-$ *is* the frequency splitting, conformally packaged. The experimental exclusion of real quantum mechanics is read as direct support: one circle and one clock supply the single shared i that experiment demands. What remains open is named exactly; why $\mathbb{C}$ is beautiful mathematics is conceded to lie beyond physics.

Executive Summary

The question. Since Cardano's cubics, complex numbers have been mathematics' most productive invention; since 1925 they have been physics' most puzzling necessity. Penrose has posed the question at full strength for fifty years: why should nature run on a number system invented for algebra — in its quantum amplitudes, and, most sharply, in the holomorphic geometry of twistor theory, where massless fields become cohomology on a complex three-space and the split into positive and negative frequency becomes the geography of that space?

The answer defended here. Nature does not run on complex numbers. It runs on two globally shared *real* structures — a compact, non-spatial phase circle (the sixth dimension of the GG-6 bulk, forced by Theorem 0 of the parent program) and a causal clock with energy positivity (AX-1 and AX-3 of its constitution) — and every complex structure of physics is their derived packaging. Eight theorems carry the claim. T-P4-1: the gauge phase of charged matter is circle geometry — charge quantization its compactness, the Aharonov–Bohm phase its holonomy, superselection its unreadability, the quarter-turn $J_Q = \partial_\theta / Q$ its unique complex structure. T-P4-2: the i of the state space is the polar phase of the time-translation generator — concretely the Hilbert transform, a real integral operator — unique by positivity on the static arena. T-P4-3: the two coincide on charged sectors exactly under the Feynman–Stückelberg pairing, so *antiparticles are the consistency condition between the circle and the clock.* T-P4-4: one shared J makes the complex tensor product a quotient of the real one — the structure the network Bell experiments demand. T-P4-5: the discrete-symmetry table is orientation bookkeeping (antiunitary iff T is involved), and CP violation is chirality of loop holonomies, lattice-locked on the pentagonal alphabet of the integer ledger. T-P4-6 and T-P4-7: positive frequency *is* holomorphy — Hardy space with one clock, the forward tube with all clocks, and twistor space at conformal symmetry, whose division $\mathbb{PT}^+/\mathbb{PT}^-$ is the frequency splitting itself. T-P4-8: conditionally on the standard scattering hypotheses, the interacting theory's i is the free i of its asymptotic frame.

The apparatus. The paper is built to be checked rather than believed: a five-century history with its chronology; a preliminaries Part defining every object from scratch; three graduate courses (analysis, twistor geometry, and the two sources) totaling nineteen lectures with problem sets; six worked examples, from the analytic signal to horizon thermality to a CKM plaquette computed with no complex arithmetic; four figures and five tables, including a twelve-aspect structural comparison with twistor theory and a three-roads comparison that adds string theory; forty-two answered objections from seven referee communities; and seventy-nine references, each carrying a paragraph-length note on what it established and how it is used.

Philosophical remarks. Three boundaries are drawn with care. *Invention and discovery:* the historical sequence — algebraic invention (1545), geometric domestication (1797–1831), physical indispensability (1925), bafflement (1960–) — resolves as a species discovering the compressed format three centuries before the thing compressed; the structure was found, the notation was invented, and the felt magic was the gap between the two dates. *The isomorphism concession:* the claim is not that nothing isomorphic to $\mathbb{C}$ exists in nature — the circle with its rotation *is* such a structure, which is exactly why the notation works — but that no complex primitive is instantiated:

the direction of explanation runs from geometry to notation, never back. *The Platonic boundary:* everything here concerns nature's use of complex numbers; why $\mathbb{C}$ is beautiful *mathematics* — algebraic closure, Cauchy rigidity, the shortest paths between real truths — is a question no physics can close, and it is left, intact and with respect, where Penrose keeps it. On this account Wigner's unreasonable effectiveness loses one of its sharpest instances not by dissolution but by identification: the effectiveness was never unreasonable, only unrecognized — the mathematics was a portrait, and the sitter has now been named.

Who should read it, and how. *A twistor or mathematical-relativity reader* meets a duality claim stated in their own theorems — two of the bridge's three levels are the school's own — with the direction of explanation reversed and the googly problem honestly marked open on both sides. *A particle-theory graduate student* meets a self-contained course sequence connecting Hilbert transforms, discrete symmetries, CP phases, and twistor space to one geometric source, with problem sets designed for a rainy afternoon (see the Preface). *A quantum-foundations reader* meets the real-Hilbert-space exclusion experiments turned from a threat into the sharpest support, and the reconstruction theorems placed as convergent evidence of a different kind. *A general physicist* meets a plain thesis with named falsifiers: the cosmos carries a circle, a clock, and orientations — all real — and complex numbers are how four-dimensional observers compress what they cannot see. One boundary deserves its own sentence: the interacting massless infrared — the corner where this account was weakest and where the twistor school keeps its deepest commitments — is crossed at the rigorous-model level by the door theorem (T-P4-9): the soft-cloud multiplicity is port bookkeeping at the operational dictionary, and the derived i passes through every sector untouched. With charge and the horizon vacua this completes a custody principle met three times on three strata: the dictionary owns the sectors; the geometry owns the i. The register throughout is the cohort's calibrated one: derived where provable, conditional where conditional, open where open, and conceded where concession is the honest coin.

Keywords: complex numbers; positive frequency; Hilbert transform; Hardy space; analytic signal; Paley–Wiener theorem; forward tube; Wightman analyticity; twistor theory; Penrose transform; cohomology; holomorphy; $\mathbb{PT}^+/\mathbb{PT}^-$; gauge phase; charge quantization; Aharonov–Bohm; superselection; antiparticles; Feynman–Stückelberg pairing; CPT; discrete symmetries; CP violation; Jarlskog invariant; pentagonal lattice; BI-6 integer ledger; flavor holonomy; infraparticle; infrared sectors; horizon thermality; ports; polar phase; one-particle structure; real presentation; shared complex structure; network Bell tests; GG-Theory; sixth dimension; phase circle; scale dimension; OGGEP; GDOS; GRIP; philosophy of mathematics; invention versus discovery; unreasonable effectiveness; falsifiability

Nothing Was Invented; Something Was Found — and What Was Found Was a Circle and a Clock

Where the complex numbers of physics come from. Above: the two real sources at the top of the page — one compact, non-spatial phase circle and one causal clock with energy positivity — and the single complex structure they force, read at three levels of symmetry. With one clock it is Hardy-space holomorphy and the imaginary unit is the Hilbert transform, a real integral operator. Demanding agreement among all inertial clocks enlarges it to the forward tube. Demanding it under the full conformal group completes it to twistor space. Below: that top level drawn — projective twistor space cut in two by the sign of the helicity form, so the split between positive and negative frequency becomes the geography of a complex three-space. Two of the bridge's three levels are the twistor school's own theorems, credited as such; what is claimed here is the direction of explanation, and only that. In full: Figure 4, p. 43, and Figure 3, p. 42.

the two real sources (G1, G2)
one compact phase circle; one causal clock with positivity
$\Rightarrow$ the derived complex structure J (T-P4-1–T-P4-3)

$\downarrow$

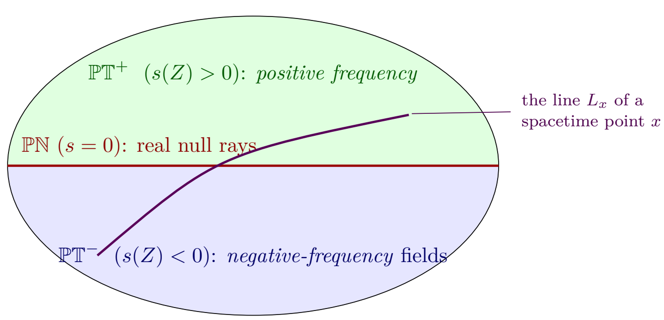
Hardy level (one clock; T-P4-6)
positive frequency $\Leftrightarrow$ holomorphy in the upper half of complexified proper time; J = Hilbert transform = the splitting's generator

$\downarrow$ enlarge to Poincaré

Tube level (all inertial clocks)
one imaginary direction per timelike direction: the forward tube $\mathcal{T}^+$; Wightman analyticity as causal positivity's field form

$\downarrow$ enlarge to conformal (massless)

Twistor level (all conformal clocks; T-P4-7)
the tube's conformal completion: massless helicity-h fields as $H^1(\mathbb{PT}^{\pm}, \mathcal{O}(-2h-2))$; the division $\mathbb{PT}^+/\mathbb{PT}^-$ *is* the frequency splitting



on any timelike worldline the division restricts to the Hardy splitting of T-P4-6: the twistor geometry is the frequency dial, drawn at conformal symmetry

Overview of Contents

List of Figures

List of Tables

Contents

Preface

This paper is written to be readable, end to end, by a graduate student in particle theory at a major university — and it was structured so that such a reader loses nothing by knowing no GG-Theory in advance. Everything imported is restated with its hypotheses; every definition used is made in Part II; every theorem is proved or proof-sketched on the page, with the physics said in words immediately after the mathematics.

Here is the honest pitch for spending a week with it. You already own the pieces: the analytic signal from your signals course, the Hilbert transform from complex analysis, positive frequency from field theory, the C, P, T table from your first QFT semester, and (perhaps) the Penrose transform from a differential-geometry elective. What you likely do not own is the statement that these are *one* piece: that the operator splitting frequencies is the unique complex structure causality and positivity allow; that the antiparticle pairing is where a compact circle and a causal clock agree to share it; that the C/P/T table is orientation bookkeeping of three real factors; and that the celebrated division of twistor space is this same splitting wearing conformal clothes. Those identifications are theorems here, each about a page long, and each checkable with tools you already have.

A suggested week. Days one and two: Part I for the story and Part II for the definitions — do not skip the definitions; the whole paper is linear algebra once they are owned. Day three: Part IV's five theorems, with Course C open beside them for the drills. Day four: the bridge (Part V) with Course A as companion — the analytic signal by hand in the morning, the tube in the afternoon. Day five: Course B straight through to Lecture B.4, which is the single derivation this paper most wants you to own: one line of spinor algebra putting a complexified point's twistor line in $\mathbb{PT}^+$. The weekend: worked examples V and VI (CP violation and the Majorana door, both with no i anywhere), then the comparison tables and as many of the forty-two objections as you have appetite for — they are arranged so that your own objection is probably among them.

Three further notes. The problem sets carry hints and answers because they are meant to be *done*, not admired; none needs more than a page. The annotated reading guide (Part IX) re-shelves the bibliography into five ordered paths, so the week can become a semester if the subject takes hold. And the register of the paper — what is proved, what is conditional, what is open, what is conceded — is itself part of the training: foundations work is only worth doing with the books kept this way, and a student who internalizes the ledger discipline will keep it in whatever field they land. Whether or not you ever adopt the parent framework, the identifications are yours to keep; physics has always advanced by noticing that two familiar things were one thing, and this paper is an extended exercise in exactly that noticing.

Part I
The Question and Its History

Five centuries of complex numbers, in mathematics, in physics, and in Penrose's hands

1 Part I in plain English: the question, and why five centuries did not close it

Complex numbers entered mathematics as an embarrassment and left it indispensable, and then the same thing happened again in physics. The phasor was a bookkeeping convenience that could be discarded at the end of the calculation. The gauge phase could not be discarded. The quantum amplitude could not be discarded either, and in 2021 an experiment excluded the version of quantum mechanics that tries.

What none of that supplies is a reason. Penrose put the question at full strength — why is nature's massless physics holomorphic? — and answered it by making holomorphy a primitive endowment. That is an answer of a particular kind. It declines to look behind the complex numbers, on the ground that there is nothing behind them, and it has been the best answer available for sixty years.

This Part tells the history in three passes, mathematical, physical and quantum-mechanical, states Penrose's question in his own terms, and then states what this paper will claim against it. Nothing is derived here; the Part also lays out the two maps a reader will want later, of the cohort and of the eleven Parts. One boundary is drawn at the outset and held throughout: the claim is about nature's use of the complex numbers, not about why they are beautiful mathematics, which is a question no physics can close.

2 An invention that would not stay invented: the mathematical history

Complex numbers entered mathematics with an apology. Cardano's *Ars Magna* (Cardano, 1545) met square roots of negative quantities midway through the solution of cubic equations and called the work of keeping them "as subtle as it would be useless"; Bombelli (Bombelli, 1572) showed the subtlety pays — real roots of real cubics sometimes *require* passage through the impossible quantities — and for two centuries the new numbers lived as bookkeeping: tolerated in the middle of a calculation, banished from its ends. The turn came when the algebra found its geometry. Euler bound the exponential to the circle (Euler, 1748); Wessel (1797), Argand (Argand, 1806), and Gauss (Gauss, 1831) laid the numbers on a plane, making i a quarter-turn rather than a mystery; and in the nineteenth century Cauchy (Cauchy, 1825) and Riemann (Riemann, 1851) discovered that functions of the new variable are not a generalization of real analysis but a crystallization of it — differentiability once means differentiability forever, integrals depend on nothing but topology, and a function's values in any small region determine it everywhere. By 1900 the complex plane had become what it remains: the most rigid, most consequence-rich structure in analysis. The invention had turned out to be a discovery; the only question was of what.

Two further episodes of the mathematical story matter here. Hamilton, disturbed that analysis should rest on an "impossible" quantity, rebuilt the complex numbers as ordered *pairs* of reals with a stipulated multiplication (Hamilton, 1837) — the first real presentation, and philosophically decisive: it proved i eliminable in principle. It changed no practice, because nothing then selected the pairing rule; his subsequent quaternions showed that other rules exist, deepening the question of why nature (if it chose at all) chose this one. And the rigidity discovered by Cauchy, Riemann, and Weierstrass — one complex derivative implying all of them, values in a disc determining values everywhere — marked complex analysis as qualitatively unlike real analysis: not a bookkeeping convenience but a crystalline structure. Both episodes are answered rather than merely admired in what follows: the pairing rule is selected by a circle and a clock (Part IV), and the rigidity is the analytic face of one-sided frequency support (Part V).

3 The quiet infiltration of physics

Physics adopted the complex numbers in three waves, each time protesting that the adoption was mere convenience. The first wave was oscillation: from Fourier's analysis onward, and canonically in Steinmetz's phasors, every wave problem was compressed by writing the real oscillation $\cos \omega t$ as the real part of $e^{i\omega t}$ — convenience, plainly, since the physics stayed real and the i was discarded at the end. The second wave was gauge structure: Weyl's 1929 reinterpretation of his failed scale invariance (Weyl, 1918) as a local *phase* invariance (Weyl, 1929) put $e^{i\lambda(x)}$ into the heart of electrodynamics, and the modern bundle formulation (Wu and Yang, 1975) recognized the phase as holonomy — with the Aharonov–Bohm effect (Aharonov and Bohm, 1959) confirming that the holonomy, and only the holonomy, is physical. The third wave was the quantum revolution, and it was different in kind: this time the i refused to be discarded.

The first wave deserves one more sentence of respect, because its very success set the trap. Fresnel's complex amplitudes for light (Fresnel, 1823), Fourier's exponentials (Fourier, 1822), Steinmetz's phasors for alternating current (Steinmetz, 1893): in every case the recipe was identical — complexify, compute, discard the imaginary part — and in every case it worked because a linear oscillation is two real quantities (amplitude and phase, or cosine and sine components) that the complex symbol carries as one. Physics thereby trained itself for eighty years to regard i as removable scaffolding. When quantum mechanics arrived with an i that was *not* removable, the training became the puzzle: the tools gave no way to say what had changed. Parts IV and V give the missing statement: the phasor's i compresses one mode's cycle; the quantum i is the shared quarter-turn of all modes — an orientation, not a shorthand.

4 Quantum mechanics: the wave that would not stay real

Schrödinger did not want the i (Schrödinger, 1926). His first instinct was that the wavefunction was a real vibration of something; the complex unit in $i\hbar\,\partial_t\psi = H\psi$ entered under protest, and the discomfort — a fundamental equation of nature built on a number invented for algebra — never fully left the subject. Three facts hardened the discomfort into a puzzle. First, the complexity is not notational: relative phases interfere, and the phase of an amplitude is measured (as always, through holonomies and rates) in every double-slit and every Ramsey fringe. Second, the complexity is not optional: quantum mechanics rebuilt over real Hilbert spaces (Stueckelberg, 1960) with real tensor products makes different predictions in network Bell configurations, and experiment has excluded it (Renou *et al.*, 2021) — nature insists on a single complex structure shared by independent sources. Third, the complexity is universal: a strictly neutral field, with no gauge phase at all, needs the quantum i exactly as much as a charged one. Whatever the i of quantum mechanics is, it is not the $e^{i\omega t}$ convenience of the phasor engineer.

The subsequent history sharpened the necessity without explaining it. Born's rule (Born, 1926) made $|\psi|^2$ physical while leaving the phase of ψ unmeasured — yet not unphysical, since *relative* phases drive every interference pattern; Dirac's transformation theory (Dirac, 1930) wove the i into the commutation relations themselves; and the modern era made the geometry explicit piece by piece: the Aharonov–Bohm effect showed the electromagnetic phase to be a loop quantity, Berry's phase (Berry, 1984) showed adiabatic transport to be holonomy, and quantum information rebuilt the whole formalism operationally without ever dislodging the complex field. By 2021 the situation was experimentally clean and conceptually strange: the real-Hilbert-space alternative was falsified in the laboratory (Li *et al.*, 2022; Renou *et al.*, 2021), so nature demonstrably uses a single shared complex structure — and no principle in the textbook said what it was shared *by*. That is the precise gap this paper fills: the shared structure has two real sources, one circle and one clock, and the experiment is their fingerprint.

5 Penrose, twistors, and the question at full strength

No one has posed the question with more force, or higher stakes, than Roger Penrose. In *The Road to Reality* (Penrose, 2004) and five decades of lectures he has singled out the "magic" of complex numbers as one of the deepest unexplained facts about nature — and he has raised the stakes beyond quantum amplitudes. Twistor theory (Penrose, 1967; Penrose and MacCallum, 1973) proposes that spacetime physics itself is best written on a complex three-dimensional projective space $\mathbb{PT}$, where the free massless fields of every helicity become *holomorphic* objects — cohomology classes, in the rigorous formulation (Eastwood *et al.*, 1981) — and where, remarkably, the split of field into positive and negative frequency, so labored in spacetime language, becomes the geometric division of $\mathbb{PT}$ into two halves. For Penrose this is evidence that complex holomorphy is not descriptive apparatus but constitutive of nature. The question at full strength is therefore double: why do quantum amplitudes live in $\mathbb{C}$, and why do massless fields live on complex projective space? A complete answer owes him both halves.

The origin of the twistor idea bears on how this paper engages it. Penrose reached twistors from a conviction (already visible in his spin-network work) that continuum spacetime should be derivative and combinatorial-geometric structure primary; the 1967 construction realized light rays — not points — as the primitive objects, with the Robinson congruence supplying the picture of a twistor as a twisting family of rays and the helicity form its invariant. The program's subsequent half-century divides naturally: the classical period (1967–1981) built the transform and its cohomology; the long middle kept the geometry alive against the field's momentum; and the amplitudes renaissance from 2004 onward (Witten, 2004) returned twistor variables to the center of perturbative gauge theory. Through every period, one conviction has been constant — that the holomorphy is telling us something about nature, not about our bookkeeping. This paper agrees, and locates the something.

6 Five centuries in one table, and the detail behind it

Three episodes in the table deserve a paragraph more than the survey sections gave them, because each is a rehearsal of this paper's thesis.

Bombelli and the casus irreducibilis. The scandal of 1572 was not that imaginary numbers could be written down; it was that they could not be avoided: for a cubic with three real roots, Cardano's formula routes *every* algebraic path to those real answers through complex intermediaries. Real in, real out, complex in between — the exact arc this paper finds in physics (real bulk, complex amplitudes, real records), discovered four and a half centuries early as a fact about polynomials.

Hamilton's ordered pairs. In 1835 Hamilton, disturbed that algebra should rest on an "impossible" quantity, rebuilt $\mathbb{C}$ as ordered pairs of reals (a, b) with a stipulated multiplication — precisely the real presentation of Proposition 10.3. The construction was received as a philosophical nicety: it showed i was eliminable in principle, and changed nothing in practice, because nothing then *selected* the pairing rule. The present paper is, in one sense, the physics that was missing from Hamilton's algebra: nature supplies the selection (a circle, a clock, a positivity clause), and the pairing rule stops being a stipulation.

Schrödinger's reluctance. The founder of wave mechanics resisted his own equation's i: his first four communications of 1926 work hard to keep ψ real, and his correspondence of that summer frets that a complex wavefunction cannot be a physical wave. The resistance was reasonable — every previous i in physics had been removable phasor shorthand — and its failure was the discovery, made by the formalism before it was made by any argument, that this i was of a different kind. Part IV locates the difference exactly: the phasor's i compresses one mode's cycle, while the quantum i is the *shared* quarter-turn of every mode at once — the polar phase of the clock itself, which no change of variables can cancel because it is not a variable but an orientation.

7 The claim of this preprint

This preprint assembles the answer developed inside the GG-Theory program and states it for a general mathematical-physics readership. The claim has three layers. *First*, physics contains two complex structures wearing one coat: the gauge phase of charged matter and the quantum

Table 1: A chronology of the complex numbers, from algebraic embarrassment to physical necessity to the present identification. The right column names where each episode appears in this paper. The rows run from Cardano's "as subtle as useless" to the present identification — fifteen episodes over five centuries, each with its primary source cited in place. Read downward, the middle column records the complex numbers becoming steadily harder to remove from physics while none of the episodes says where they come from, which is the question this paper takes up.

Year	**Actor**	**Episode**	**Here**
1545	Cardano (1545)	*Ars Magna*: square roots of negatives appear mid-cubic; "as subtle as useless"	§2
1572	Bombelli (1572)	the *casus irreducibilis* forces them: real roots reached only through impossible numbers	§2
1748	Euler (1748)	$e^{i\theta} = \cos\theta + i\sin\theta$: the exponential bound to the circle	§2
1797–1831	Wessel, Argand (1806), Gauss (1831)	the plane: i becomes a quarter-turn; the fundamental theorem of algebra	§2
1825–1859	Cauchy (1825), Riemann (1851)	rigidity: contour independence, analytic continuation, geometry of surfaces	§2
1835–1843	Hamilton (1837)	$\mathbb{C}$ as ordered *pairs of reals* with a rule — the first real presentation — then quaternions	§2
1822–1893	Fourier (1822), Fresnel (1823), Steinmetz (1893)	the phasor: oscillation compressed into $e^{i\omega t}$, discarded at the end	§3
1918–1929	Weyl (1918), Weyl (1929)	scale invariance fails, phase invariance succeeds: the gauge $e^{i\lambda(x)}$	§3
1925–1927	Schrödinger (1926), Born (1926), Dirac (1930)	the i that would not cancel: complex amplitudes, reluctantly	§4
1959–1984	Aharonov and Bohm (1959), Wu and Yang (1975), Berry (1984)	the phase is holonomy: geometry, measured	§3
1963–1977	Penrose (1967), Penrose and MacCallum (1973)	twistors: massless physics as holomorphy on $\mathbb{PT}$; the question at full strength	§5
1981	Eastwood *et al.* (1981)	the transform made rigorous: $H^1(\mathbb{PT}^{\pm}, \mathcal{O}(-2h-2))$	§14
2004	Witten (2004)	the twistor string: amplitudes renaissance	§43
2021	Renou *et al.* (2021)	real quantum mechanics (unshared i) experimentally excluded	§4
2026	this preprint	positive frequency = holomorphy = the derived J; the sources named	Parts IV–V

structure of the state space; they coincide for charged fields, which is why a century of quantum electrodynamics could not tell them apart. *Second*, both are derived from real structures: the gauge phase from a compact, non-spatial phase circle (the sixth dimension of the GG-6 bulk), and the state-space i from causal time orientation plus energy positivity, which select the polar phase of the time-translation generator uniquely on a static arena; their consistency on charged sectors is exactly the Feynman–Stückelberg pairing, so antiparticles are the handshake between the circle and the clock. *Third* — the layer addressed to the twistor school — holomorphy itself is the same derivation seen at large symmetry: positive frequency is Hardy holomorphy, tube analyticity is its field form, and the twistor division $\mathbb{PT}^+/\mathbb{PT}^-$ *is* the frequency splitting, conformally packaged. Nothing in this diminishes the twistor construction; it derives its motivation. And one boundary is drawn in advance: this is an answer about *nature's* use of complex numbers. Why the complex numbers are beautiful *mathematics* — algebraic closure, Cauchy rigidity — is a question no physics can close, and it is left, with respect, where Penrose keeps it.

8 The cohort inheritance and the roadmap of the eleven Parts

Before the toolbox opens (Part II) and the arena is built (Part III), it is useful to lay out two reader-orientation maps on the same opening: a panorama of the twenty-two-paper cohort in which this paper stands (§8.1, Figure 1), and a panorama of the eleven Parts of the present paper (§8.2, Figure 2). A reader of any later Part should be able to flip back to this Section and orient themselves in both directions: which cohort companion carries which imported result, and where the present argument stands in the eleven-Part arc.

8.1 The cohort inheritance

This paper is a member of the twenty-two-paper GG-Theory cohort — eleven A-papers, six P-papers, three C-papers and two white papers: it stands in the Foundations Branch, where it carries the origin of i, and it inherits the cohort's results in completed tense. Its designator, GG-P4-Real, names the thesis rather than the toolkit: the six-dimensional bulk is real, and the complex unit that physics observes is manufactured — derived, never primitive. Twistor geometry keeps every page here that it has earned, as the sharpest developed contrast: Penrose's program takes holomorphy as the primary fact, and this paper derives that same holomorphy from two real structures. The constitutional frame — four axioms, of which the first (locality–causality) and third (the operational dictionary with its positivity clause) supply this paper's two selection principles — is Arisaka, 2026, A1-Constitution. The arena — four-dimensional spacetime plus a non-spatial scale interval and a non-spatial, compact phase circle, with a globally static metric — is Theorem 0 of Arisaka, 2026, A2-GG6; the integer ledger behind the lattice-locked phases quoted below is Arisaka, 2026, A3-BI6; the geometrization lineage from Euclid to the present is traced in Arisaka, 2026, A4-Euclid. The boundary law (the open-system projection whose partial trace will reappear here at horizons and in the infrared) is Arisaka, 2026, A5-GDOS; its geometric backbone is Arisaka, 2026, A6-GRIP; its information backbone is Arisaka, 2026, A7-Qi4. The quantum-mechanics treatment whose real-bulk series (RB-1–RB-12) this preprint expands and re-derives for a broader readership is Arisaka, 2026, A8-QM; the Standard-Model consequences — the derived discrete-symmetry table, the flavor holonomy theorems, the pentagonal CP phase — are Arisaka, 2026, A9-SM; the cosmological consequences, including the dial-free-door theorem that forces the Majorana channel, are Arisaka, 2026, A10-Cosmos; the philosophical summit is Arisaka, 2026, A11-Origin. Of the phenomenology branches, the non-spatial-circle contrast with Kaluza–Klein is sharpest in Arisaka, 2026, P1-KK; the neutrino count in Arisaka, 2026, P2-N3; the electron's ground-state reading in Arisaka, 2026, P3-Electron; the proton's, and with it baryon number carried as a winding of the same compact circle this paper derives the gauge phase from, in Arisaka, 2026, P6-Proton; and the three observational companions are Arisaka, 2026, C1-Kaxion, Arisaka, 2026, C2-Hubble, and Arisaka, 2026, C3-BH, the last supplying the horizon-as-port reading used in Part VI. Two white papers stand outside the numbered series and both bear on this one. Arisaka, 2026, W1-Symmetry walks the century of symmetry and conservation in the order it was found, and arrives where this paper's winding integrality already stands: at an integer with nowhere to move. Arisaka, 2026, W2-Holonomy turns on a single exact statement — a holonomy requires a closed loop, and the interval fiber has none — which is the asymmetry that lets the circle of Part IV carry a gauge phase at all, put to work there on sixteen open questions rather than on the origin of i.

The Foundations Branch. On the map's left flank stands the branch this paper belongs to: two member papers whose shared subject is the boundary's complex structure and its mirrors, a single column beside the spine. The partner paper, GG-P5-CPT (Arisaka, 2026, P5-CPT), consumes the derived complex structure one step further along: from the orientations of the same real factors whose two global structures this paper turns into the i — the compact circle, the spatial slice, and the causal clock — it derives C, P, and T as orientation reversals, the CPT theorem as total orientation reversal, the pentagon-locked CP phases, and the sign of the cosmic matter excess. On the map it is the GG-A9-SM–GG-A10-Cosmos interface: fed by the ledger phases derived in GG-A9-SM — the PMNS Dirac phase and the Majorana parity $\alpha_{31} = \pi$ — and feeding the η_B chain of GG-A10-Cosmos at the grade recorded there, derived-conditional on a named finite-temperature hypothesis; the solid shared-i edge descending from this paper's node carries what it inherits from here — one J for the whole tower, and the custody principle both papers state in the same words: *the dictionary owns the sectors; the geometry owns the i.* Both branch members carry P-series designators; their place at the spine's flank, not the phenomenology rows, marks their foundational role.

A reader needs none of these in hand: every imported statement is restated here with its hypotheses, and the derivations of Parts IV–V are self-contained at the level of statements and proof sketches. The twenty-two-paper map, identical in every paper of the cohort, is Figure 1, which closes this subsection.

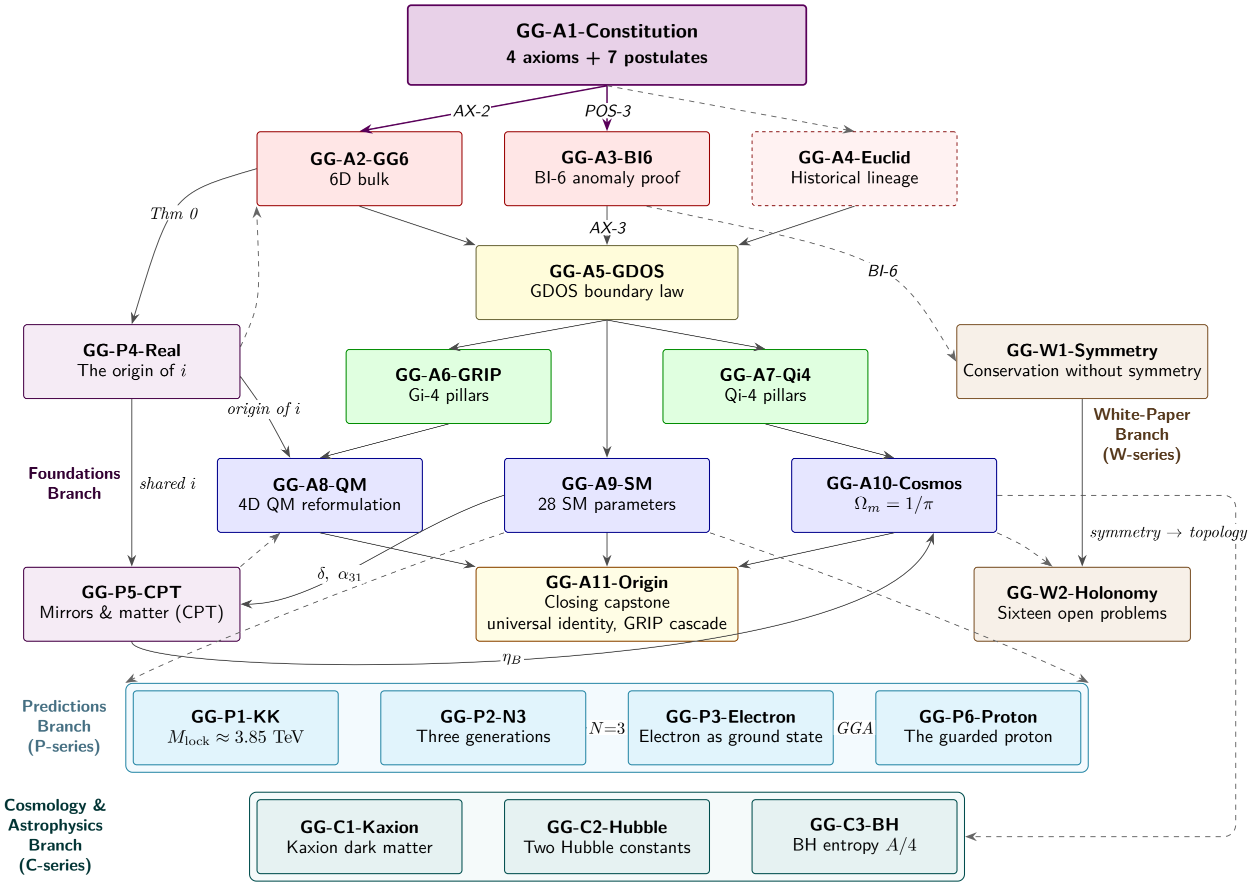


Figure 1: The twenty-two-paper cohort map of GG-Theory. The constitutional root GG-A1-Constitution (violet, top) supplies the four axioms and seven postulates from which the rest of the cohort descends: the GG6 Trio (GG-A2-GG6, GG-A3-BI6, GG-A4-Euclid, red) realizes the bulk geometry; GG-A5-GDOS (yellow) the boundary law; the GDOS Trio pillars (GG-A6-GRIP, GG-A7-Qi4, green) the geometric and quantum-information runtime; the 4D-Physics Trio (GG-A8-QM, GG-A9-SM, GG-A10-Cosmos, blue) the textbook readouts; and the closing capstone GG-A11-Origin (gold) the quarks-to-consciousness identity cascade. Two framed branches carry the phenomenology — the **Predictions Branch** (cyan), whose four papers hand their results along the row by the internal arrows (GG-P2-N3's N=3 into GG-P3-Electron, and that paper's ground-attractor method, GGA, on into GG-P6-Proton), and the **Cosmology & Astrophysics Branch** (teal). The spine is flanked by two columns of two, which do opposite jobs. On the left the **Foundations Branch** (violet) feeds it: GG-P4-Real, fed by GG-A2-GG6's Theorem 0 arena, manufactures the i that GG-A8-QM consumes, and GG-P5-CPT is the GG-A9-SM–GG-A10-Cosmos interface, fed by the ledger phases and feeding the η_B chain across the width of the map, with the shared-i edge descending solid between them. On the right the **White-Paper Branch** (brown) reads it: GG-W1-Symmetry walks the century from Noether to Hopkins–Singer, and GG-W2-Holonomy dissects the structure that arc arrives at — protection by topology rather than by symmetry — across sixteen open problems. The white papers derive nothing new, carrying every result at its parent's declared grade; the single dashed feed drawn into each names its heaviest source, not its only one. *Reading conventions:* solid arrows mark axiom-forced descent and carry an edge label naming the driving axiom or postulate where most illustrative; dashed arrows mark the deployment of derived results, or historical links inherited without being constitutionally forced.

8.2 Roadmap of the eleven Parts

The rest of the paper is organized into ten further Parts, and the walk through them is plotted in Figure 2. The present Part — the question and its history — is nearly done: five centuries of the complex numbers, their quiet infiltration of physics, Penrose's question at full strength, the claim of this preprint, and the cohort the paper stands in. Everything after this page is that claim made precise, and the itinerary below says where each piece of the argument lives.

Two Parts build the arena. Part II opens the toolbox — real symplectic data, complex structures, the polar phase of a generator, Hardy spaces, and twistor space — with every object defined from scratch, so that the argument can be checked rather than believed. Part III then builds the stage on which those tools act: the three constitutional clauses this paper uses, the six-dimensional geometry that Theorem 0 forces from them, and the two globally shared real structures — a circle and a clock — that the rest of the paper never leaves.

Parts IV and V are the heart. Part IV derives the two real sources and their consistency condition: the gauge phase from the circle, the state-space i from the clock, the antiparticle handshake where the two meet, the shared complex structure that the handshake enforces, and the discrete-symmetry table that falls out of it. Part V — highlighted in the roadmap, because it is what the title claims — proves the bridge: positive frequency is holomorphy, at one clock (Hardy space), for fields (the forward tube), and at full conformal symmetry (the twistor division of $\mathbb{PT}$), with the consequence for twistor theory drawn on the spot.

Part VI takes the derivation to the places where derivations usually break: interacting matter and the asymptotic completion, the infrared door (sectors are ports, and the derived J is sector-blind), and the three superselections the program meets, unified under one custody principle.

Two Parts then weigh the case. Part VII discusses significance and uniqueness, sets the three roads side by side — string theory, twistor theory, GG-Theory — states what is claimed and in what regime, and closes with the predictions for falsification and the open questions. Part VIII states the conclusions: what the paper has established, the objective limitations, and the vision of a century in which number systems are read as geometry.

The back matter is the workshop. Part IX collects the reference dictionaries and the pocket course — from the analytic signal to the Penrose transform, at lecture depth. Part X answers the common objections, community by community, including the ones a twistor theorist should raise; and Part XI closes the paper with the annotated references, each carrying a note on what it established and how it is used here. Read in order, the eleven Parts compose into a single argument; read by need, each Part names its imports and can stand a visit on its own.

Close of Part I, in plain terms. Five centuries ago a number was invented that no one wanted; a century ago it became indispensable to physics; fifty years ago Penrose asked why, at full strength, for amplitudes and for the holomorphic geometry of light alike. This Part told that story and stated this paper's answer in outline: the complex numbers are the compressed notation for two real structures — a circle and a clock — that a four-dimensional observer cannot see directly. Everything that follows is the outline made precise. The next Part lays out the mathematical tools — none of them exotic, all of them defined from scratch — with which the claim can be checked rather than believed.

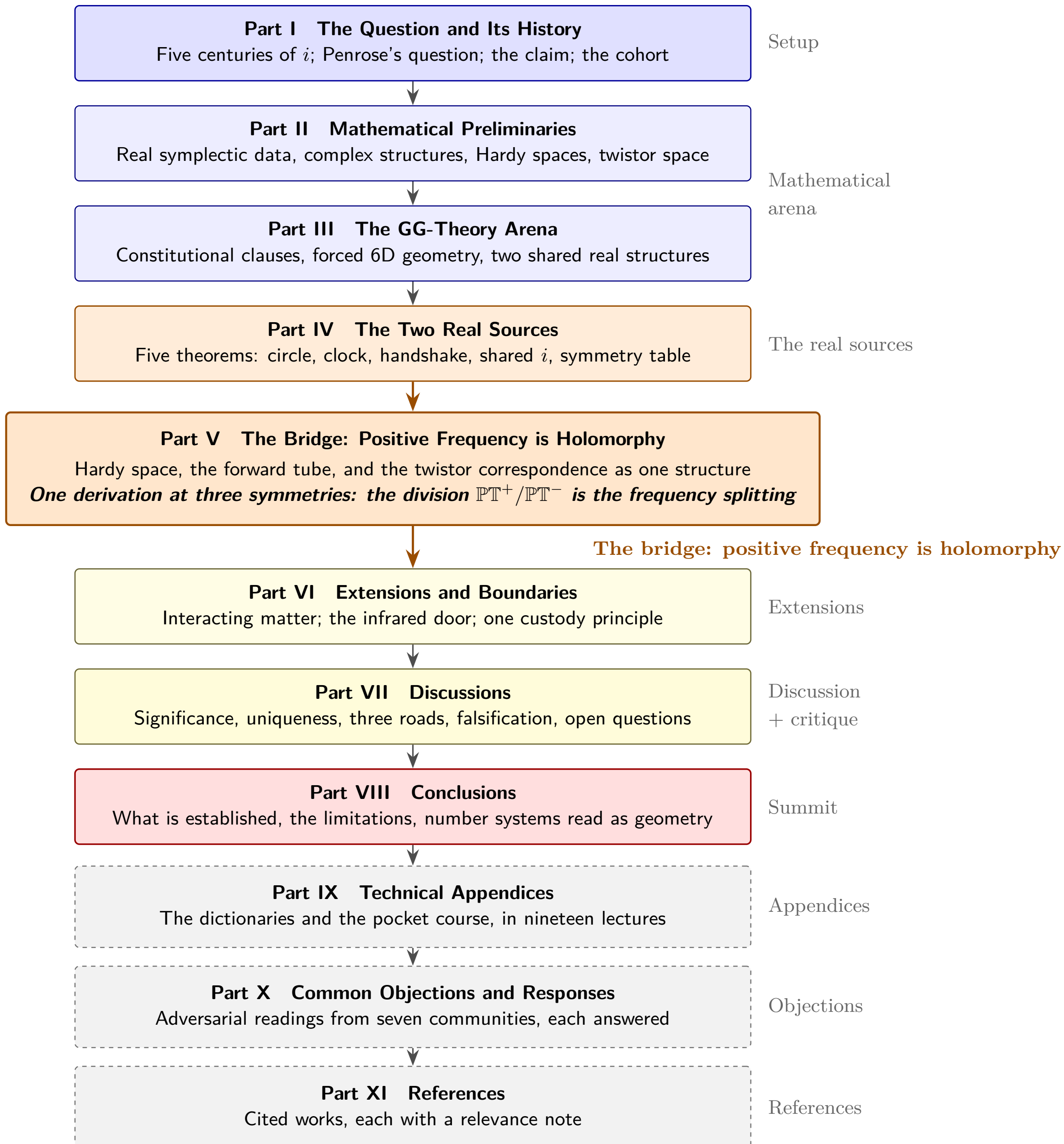


Figure 2: Roadmap of the eleven Parts of GG-P4-Real. The argument develops in phases. *Setup* (Part I) tells the five-century history of the complex numbers, states Penrose's question at full strength and this paper's claim, and locates the paper in its twenty-two-paper cohort. The *mathematical arena* (Parts II–III) defines every object the argument uses from scratch and builds the stage: the constitutional clauses, the forced six-dimensional geometry, and the two globally shared real structures — the circle and the clock. The *real sources* (Part IV) derive the gauge phase from the circle and the state-space i from the clock, with the antiparticle handshake as their consistency condition. The *bridge* (**Part V**, highlighted) is the central payoff: positive frequency is holomorphy, proved at the Hardy, tube, and twistor levels. *Extensions* (Part VI) carry the derivation to interacting matter, the infrared door, and the three superselections under one custody principle. Part VII discusses significance, uniqueness, the three roads, and the falsifiers; Part VIII states the conclusions; and the three back-matter Parts collect the technical appendices and the pocket course (Part IX), the common objections and responses (Part X), and the annotated references (Part XI). Part V is highlighted because the bridge is what the title claims, and the Parts before it exist to prove it.

Part II
Mathematical Preliminaries

Real symplectic data, complex structures, the polar phase, Hardy spaces, and twistor space — everything the argument stands on, defined

This Part fixes definitions and records, with proofs where they are short and with precise citations where they are classical, every mathematical object the argument uses. A reader fluent in one-particle structures, Hardy spaces, and the Penrose transform may skim to Part III; a reader meeting these objects for the first time will find here everything needed to check the later Parts line by line.

Conventions. Throughout: units $\hbar = c = 1$; metric signature $(+,-,-,-)$, as implied by the Hermitian-matrix form (40) of a spacetime point; two-component spinor indices raised and lowered with the antisymmetric ϵ_{AB}, $\epsilon_{A'B'}$ in the north-west/south-east convention of Penrose and Rindler (1986); complex conjugation exchanges primed and unprimed indices; and the helicity form carries the factor $s(Z) = \frac{1}{2} Z^\alpha \bar{Z}_\alpha$ of (9). Real presentations use the block dictionary of Proposition 10.3, with complex conjugation as block transpose.

9 Part II in plain English: the toolbox, opened before it is used

Nothing is proved in this Part. It defines every object the later Parts use — real symplectic data and compatible complex structures, the polar phase of a generator, weight decomposition under a circle action, the Hilbert transform and Hardy space, twistor space and the Penrose transform — with proofs where they are short and precise citations where they are classical.

The Part exists because the entire content of the argument is a direction of explanation, and a direction can only be argued if both ends are pinned down first. Everything the later Parts import is therefore stated here at the strength it will be used at and nowhere stronger.

The one definition worth slowing down for is the polar phase. It is elementary — an antisymmetric generator, its modulus, and the unitary factor of the decomposition — and it is the object the central theorem turns on. If it looks too small to carry that weight, the reaction is a fair one and worth keeping until Part IV.

10 Real symplectic data and complex structures

Definition 10.1 (Real phase-space data)**.** A *real symplectic space* is a pair (V, σ) with V a real vector space and σ a non-degenerate antisymmetric bilinear form. A *real Euclidean space* (the fermionic case) is a pair (V, s) with s symmetric and positive-definite. In field theory V is the real solution space of a linear field equation; for a Klein–Gordon field with Cauchy data (φ, π),

$$\sigma\big((\varphi_1, \pi_1), (\varphi_2, \pi_2)\big) \;=\; \int_\Sigma (\varphi_1\, \pi_2 - \varphi_2\, \pi_1)\, d^3x, \tag{1}$$

which is real, antisymmetric, and contains no complex ingredient.

Definition 10.2 (Complex structure; one-particle structure)**.** A *complex structure* on (V, σ) is a real-linear operator $J : V \to V$ with

$$J^2 = -\mathbf{1}, \qquad \sigma(Jf, Jg) = \sigma(f, g), \qquad \sigma(f, Jf) > 0 \;\; (f \neq 0), \tag{2}$$

the three conditions being called squaring, compatibility, and taming. On a Euclidean (V, s) the taming condition is replaced by orthogonality, $s(Jf, Jg) = s(f, g)$, plus positivity of the induced one-particle energy. A *one-particle structure* is the pair (V, σ, J) (respectively (V, s, J)); it defines a complex Hilbert space by completing V in the inner product

$$\langle f, g\rangle \;=\; \tfrac{1}{2}\,\sigma(f, Jg) \;+\; \tfrac{i}{2}\,\sigma(f, g), \tag{3}$$

with multiplication by i implemented by J. Every complex Hilbert space of physics arises this way from real data plus a choice of J; the entire content of this preprint is that in GG-Theory the choice is not free — J is constructed, uniquely, from real geometry.

Proposition 10.3 (The real presentation of $\mathbb{C}$)**.** *Let* $J_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *on* $\mathbb{R}^2$*. Then* $(\mathbb{R}^2, J_0) \cong \mathbb{C}$ *as complex vector spaces via* $(a, b) \mapsto a + ib$*; a complex number* $z = a + ib$ *acts on* $\mathbb{R}^2$ *as the rotation-scaling block*

$$B(z) \;=\; \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;=\; |z|\, R(\arg z), \tag{4}$$

and complex conjugation corresponds to the transpose $B(z)^{\mathsf{T}} = B(\bar z)$ *— i.e. to orientation reversal of the plane.*

Proof. Direct computation: $B(z)B(w) = B(zw)$, $B(z)(a', b')^{\mathsf{T}}$ matches $z \cdot (a' + ib')$ under the identification, and $B(z)^{\mathsf{T}} = B(\bar z)$ by inspection of (4). ∎

Physics intuition. Nothing in quantum mechanics ever required the *symbol* i: every complex amplitude is a plane vector, every phase a rotation, every conjugation a reflection. What physics requires is the *choice of rotation* — which real quarter-turn plays the role of i, the same choice on every system. The question "why complex numbers?" is therefore, exactly, the question "what selects J?" — and it is answerable because J is a real geometric object.

The three conditions of (2) deserve one further paragraph each, because each carries physics. *Squaring* ($J^2 = -\mathbf{1}$) makes J a bona fide imaginary unit: every vector sits in a two-plane that J rotates by a quarter-turn, so the real space decomposes into "complex lines." *Compatibility* makes those rotations canonical transformations: multiplying by a phase is a symmetry of the classical structure, which is why global phases can be unobservable at all. *Taming* is the positivity that turns the pair (σ, J) into a Kähler structure — the inner product (3) is positive-definite exactly because of it — and it is the one condition with a thermodynamic smell: it is where "energy is bounded below" enters the kinematics. A triple (V, σ, J) satisfying all three is, in the geometer's language, a linear Kähler space; quantum mechanics' state space is the completion of exactly such a triple, and the whole question of this paper is who chooses the J.

A worked micro-example. Take one classical oscillator, $V = \{(q,p)\}$, $\sigma((q,p),(q',p')) = qp' - pq'$. The taming complex structures are $J_\omega(q,p) = (p/\omega,\ -\omega q)$, one for each frequency scale $\omega > 0$ — a one-parameter family, so compatibility and taming alone do *not* fix J. What fixes it is dynamics: demanding $[J, e^{tA}] = 0$ for the oscillator's own time evolution selects ω to be the oscillator's frequency, and Lemma 11.1 below is precisely this selection stated in general. The moral, worth carrying through the whole paper: complex structures are cheap; *dynamics-commuting, positive* complex structures are unique.

11 The polar phase of a generator

Lemma 11.1 (Polar phase)**.** *Let A be a real operator on (V, σ) generating the dynamics ($\dot{f} = Af$) — throughout this paper, either a real matrix on a finite-dimensional space or one of the standard Klein–Gordon/Dirac generators on Cauchy data, for which the spectral calculus used below is classical — skew in the Hamiltonian sense ($(\sigma A)^{\mathsf{T}} = \sigma A$), with $-A^2$ symmetrizable, positive, and boundedly invertible (the gapped case). Then*

$$J_A \ :=\ \pm\, A\,(-A^2)^{-1/2} \tag{5}$$

satisfies all three conditions of Definition 10.2, commutes with the dynamics e^{tA}, and the sign is fixed by the taming condition. On a Euclidean (V, s) with A skew-adjoint the same formula gives the unique positive-energy orthogonal complex structure. In the time domain, J_A acts on solutions as the quarter-period phase advance frequency by frequency — the Hilbert transform *in the time variable (§13).*

Proof. A commutes with functions of $-A^2$, so $J_A^2 = A^2(-A^2)^{-1} = -\mathbf{1}$. Compatibility: $\sigma(J_A f, J_A g) = \sigma(f, g)$ follows since $(-A^2)^{-1/2}$ is symmetric for σA-symmetric A and A is σ-skew. Commutation with e^{tA} is immediate. On each spectral 2-plane of A with frequency ω, J_A restricts to one of the two quarter-turns; taming (equivalently positivity of $\langle f, |A| f\rangle$) selects the orientation, and selects it coherently across planes. For the Euclidean case, replace σ-skew by s-skew-adjoint throughout; uniqueness holds because a dynamics-commuting J preserves each spectral plane, where exactly one of the two quarter-turns has positive energy. ∎

Physics intuition. The generator of time translation knows two things about every mode: how fast it turns (the frequency, $|A|$) and *which way* (the orientation, J_A). The polar decomposition separates them. The imaginary unit of quantum mechanics is the second factor — the pure "which way" of time — with the speed divided out. That is why the same i serves every mode of every field: orientation is the one thing all clocks share.

Spectral picture. It clarifies the lemma to see it spectrally. On the complexification, a skew generator has purely imaginary spectrum $\{\pm i\omega_k\}$ with conjugate eigenplanes; the polar phase acts as $+i$ on each $+i\omega_k$ eigenspace and $-i$ on its conjugate — it is the operator "$i\,\mathrm{sgn}$" of the frequency, which is why the time-domain kernel is the Hilbert transform's. Uniqueness is then transparent: any dynamics-commuting J' must act within each eigenplane, where the only quarter-turns are $\pm$ the polar phase's, and positivity fixes the sign plane by plane with one inequality. Degeneracy enlarges the commutant to a unitary group but the positivity argument survives on the real form — the drill of Lecture C.2.

12 Weight decomposition under a circle action

Lemma 12.1 (Weights of the symmetric square)**.** *Let the circle act on $\mathbb{R}^2$ with weight n, i.e. by $R(n\alpha)$. Then $\mathrm{Sym}^2(\mathbb{R}^2)$ decomposes into the invariant line spanned by a^2+b^2 and the weight-$2n$ doublet $(a^2-b^2,\,2ab)$:*

$$R(n\alpha):\ \begin{pmatrix} a^2-b^2 \\ 2ab \end{pmatrix} \longmapsto R(2n\alpha)\begin{pmatrix} a^2-b^2 \\ 2ab \end{pmatrix}. \tag{6}$$

Consequently, for $n \neq 0$ the only circle-invariant symmetric self-pairing is $a^2+b^2 = |\psi|^2$ (the Dirac pairing), while for $n=0$ all pairings are invariant. In complex notation the doublet is $(\mathrm{Re}\,\psi^2, \mathrm{Im}\,\psi^2)$: the Majorana bilinear.

Proof. Write $\psi = a+ib$; under the weight-n action $\psi \mapsto e^{in\alpha}\psi$, so $|\psi|^2$ is invariant and $\psi^2 \mapsto e^{2in\alpha}\psi^2$, which is (6) in real coordinates. An invariance equation $R(2n\alpha)v = v$ for all α forces $v=0$ when $n \neq 0$. ■

Physics intuition. A circle can only forbid what it moves. Charge-carrying pairings spin under the fiber and are outlawed; the one pairing that stands still ($\psi\bar{\psi}$) is exactly the charge-conserving one. At weight zero the circle stands guard over nothing — which is why lepton number can leak there, and only there (the dial-free door of the parent program).

Representation-theoretic remark. Lemma 12.1 is the n-th symmetric-power statement of $SO(2)$ representation theory: Sym^2 of the weight-n representation decomposes as weight $0 \oplus 2n$, and only weight 0 supports invariants. The same bookkeeping at higher orders classifies every allowed coupling: a k-linear vertex among weights $n_1, \ldots, n_k$ is invariant iff $\sum n_i = 0$ — winding conservation as pure character arithmetic, drilled in Lecture C.6. Nothing in the argument uses more than the orthogonality of characters on a circle; that so much of the Standard Model's selection structure reduces to it is the ledger's point.

13 The Hilbert transform and Hardy space

Definition 13.1 (Hilbert transform)**.** For $f \in L^2(\mathbb{R}, dt)$,

$$(\mathsf{H}f)(t) \;=\; \text{p.v.}\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t')}{t-t'}\,dt', \qquad \widehat{\mathsf{H}f}(\omega) = -\,i\,\mathrm{sgn}(\omega)\,\hat{f}(\omega). \tag{7}$$

The kernel is real; H is a real operator, non-local in time, with $\mathsf{H}^2 = -\mathbf{1}$ almost everywhere on $L^2(\mathbb{R})$ — the zero-frequency obstruction is measure-zero on the line, and becomes the genuinely excluded constant mode only in finite volume. On the oscillation pair it is the quarter-period advance: $\mathsf{H}\cos\omega t = \sin\omega t$, $\mathsf{H}\sin\omega t = -\cos\omega t$ $(\omega > 0)$.

Definition 13.2 (Hardy space)**.** H^2 of the upper half-plane is the space of functions $F(z)$ holomorphic for $\operatorname{Im} z > 0$ with $\sup_{s>0}\int |F(t+is)|^2\,dt < \infty$; such F possess L^2 boundary values as $s \downarrow 0$.

Theorem 13.3 (Paley–Wiener)**.** *$f \in L^2(\mathbb{R})$ has Fourier transform supported in $[0,\infty)$ if and only if f is the boundary value of a function in H^2 of the upper half-plane (Paley and Wiener, 1934). Equivalently: the positive-frequency subspace of L^2 is exactly the boundary Hardy space, the splitting projector is $\frac{1}{2}(\mathbf{1} + i\mathsf{H})$, and the positive-frequency subspace is the $(-i)$-eigenspace of H.*

Physics intuition. "Positive frequency" sounds like a property of a Fourier table; the theorem says it is a property of *extendability*: a history has one frequency orientation exactly when it is the shadow, on the real-time axis, of something regular in imaginary time on one side. Nothing grows there — $e^{i\omega t}$ becomes $e^{i\omega t - \omega s}$ and decays — and the operator that detects this one-sided regularity is a real principal-value integral. Holomorphy is not imported here; it is what an orientation of time *is*, analytically.

The H^2 inner product, explicitly. For later use: on boundary values, $\langle F, G\rangle_{H^2} = \int \bar{F}(t)G(t)\,dt = \int_0^\infty \overline{\hat{f}_+}(\omega)\hat{g}_+(\omega)\,d\omega$ — the positive-frequency half-line carrying the whole inner product. This is the signal-level ancestor of the one-particle inner product (3): the quantum overlap of two states is a Hardy pairing of their histories, and the "mysterious" complex inner product of quantum mechanics is, mode by mode, the overlap of two analytic signals. The Poisson kernels of Course A make every statement in this section constructive.

14 Twistor space and the Penrose transform

Definition 14.1 (Twistor space)**.** Twistor space is $\mathbb{T} \cong \mathbb{C}^4$ with coordinates $Z^\alpha = (\omega^A, \pi_{A'})$ (two-component spinors), and projective twistor space is $\mathbb{PT} = \mathbb{P}(\mathbb{T}) \cong \mathbb{C}\mathrm{P}^3$. A spacetime point x corresponds to the projective line of twistors satisfying the *incidence relation*

$$\omega^A \;=\; i\, x^{AA'}\, \pi_{A'}. \tag{8}$$

The Hermitian *helicity form*

$$s(Z) \;=\; \tfrac{1}{2}\, Z^\alpha \bar{Z}_\alpha \;=\; \tfrac{1}{2}\left(\omega^A \bar{\pi}_A + \bar{\omega}^{A'}\pi_{A'}\right) \tag{9}$$

divides $\mathbb{PT}$ into $\mathbb{PT}^+ = \{s > 0\}$, $\mathbb{PT}^- = \{s < 0\}$, and the five-real-dimensional null hypersurface $\mathbb{PN} = \{s = 0\}$, whose points are the real null geodesics of compactified Minkowski space (Penrose, 1967; Penrose and Rindler, 1986).

The correspondence, in both directions. The incidence relation (8) is linear in $\pi_{A'}$, so for fixed x its solutions form a two-complex-dimensional subspace of $\mathbb{T}$ — projectively, a line

$$L_x \;=\; \{\, [\, i\, x^{AA'}\pi_{A'},\; \pi_{A'}\,] \;:\; \pi_{A'} \neq 0 \,\} \;\cong\; \mathbb{CP}^1 \;\subset\; \mathbb{PT}, \tag{10}$$

and the dictionary runs both ways: spacetime points correspond to projective lines of $\mathbb{PT}$; two points are null separated exactly when their lines intersect, the common twistor being the light ray that joins them; and a single twistor is, when $s(Z) = 0$, a real null geodesic. A twistor with $s(Z) \neq 0$ corresponds to no single real ray at all but to a *twisting congruence* of rays — the Robinson congruence, the picture that named the subject — whose handedness is the sign of s. The conformal group of compactified Minkowski space acts on $\mathbb{T}$ as $\mathrm{SU}(2,2)$: the linear maps preserving the Hermitian form (9). Conformal symmetry, awkward and nonlinear on spacetime, is therefore *linear* on twistor space — the sense in which $\mathbb{PT}$ linearizes conformal geometry, and the reason the bridge of Part V completes only when the symmetry is enlarged that far.

Helicity and the form. The name of (9) is earned dynamically: for a massless particle with momentum $p_{AA'} = \bar{\pi}_A\pi_{A'}$ and the angular momentum built from ω^A, the Pauli–Lubanski helicity evaluates to exactly $s(Z)$ (the momentum and angular-momentum expressions are displayed in Course B, Lecture B.3). The sign of the form is the particle's handedness — which is why a division of $\mathbb{PT}$ by the sign of s can carry physical information at all, and why the same division will be found, in Part V, carrying the frequency splitting.

Theorem 14.2 (Penrose transform)**.** *For each helicity h, the solutions of the free massless field equations on (conformally compactified) Minkowski space correspond naturally to sheaf-cohomology classes on twistor space,*

$$\{\textit{massless fields of helicity } h,\ \textit{positive frequency}\} \;\cong\; H^1(\mathbb{PT}^+,\, \mathcal{O}(-2h-2)), \tag{11}$$

with negative frequency carried by $\mathbb{PT}^-$ (Eastwood et al., 1981; Penrose and MacCallum, 1973). In its original contour-integral form (Penrose, 1969), the field at x is obtained by integrating a holomorphic function of homogeneity $-2h-2$ over the line (8).

The contour form, worked. Write $f(ix\pi, \pi)$ for the restriction of a holomorphic $f(Z)$ to the line (10) — the argument abbreviates the incidence substitution (8). For the scalar ($h = 0$),

$$\phi(x) \;=\; \frac{1}{2\pi i}\oint_{\Gamma\subset L_x} f\big(ix\pi,\, \pi\big)\, \pi_{E'}\, d\pi^{E'}, \qquad f \text{ of homogeneity } -2; \tag{12}$$

for helicity $h > 0$, insert $2h$ factors of π,

$$\phi_{A'_1\cdots A'_{2h}}(x) \;=\; \frac{1}{2\pi i}\oint_{\Gamma} \pi_{A'_1}\cdots\pi_{A'_{2h}}\; f\big(ix\pi,\, \pi\big)\, \pi_{E'}\, d\pi^{E'}, \qquad f \text{ of homogeneity } -2h-2; \tag{13}$$

and for $h < 0$, differentiate instead,

$$\phi_{A_1 \cdots A_{2|h|}}(x) \;=\; \frac{1}{2\pi i} \oint_\Gamma \frac{\partial}{\partial \omega^{A_1}} \cdots \frac{\partial}{\partial \omega^{A_{2|h|}}} f \Big|_{L_x} \pi_{E'}\, d\pi^{E'}, \qquad f \text{ of homogeneity } 2|h| - 2. \tag{14}$$

In every case the homogeneity of f is tuned so that the *integrand* has total homogeneity zero in π — the condition for the integral to be well defined on the projective line — and this counting is the entire origin of the exponent in (11): each π costs $+1$, the measure $\pi\, d\pi$ costs $+2$, and f must pay the balance.

Why the field equations are automatic. Differentiating under the contour,

$$\frac{\partial}{\partial x^{AA'}} f\big(ix^{BB'}\pi_{B'}, \pi\big) \;=\; i\,\pi_{A'} \frac{\partial f}{\partial \omega^A}\Big|_{L_x}, \tag{15}$$

so every spacetime derivative of the integrand inserts one factor of $\pi_{A'}$ and one ω-derivative. For (13), contracting the inserted $\pi_{A'}$ against any of the π's already present gives $\varepsilon^{A'B'}\pi_{A'}\pi_{B'} = 0$ identically, so the massless field equation $\nabla^{AA'}\phi_{A'B'\cdots} = 0$ holds with no condition on f beyond holomorphy; for the scalar, the wave operator inserts two factors of each kind and dies on the same antisymmetric contraction, $\Box\phi = 0$. The differential equation has not been solved; it has been made *vacuous* by the geometry of the parametrization. This is the precise content of the slogan that holomorphy trades equations for regularity — and the computation just performed is Lecture B.4's, in compressed form.

Elementary states, and where frequency lives. The simplest data are the *elementary states*, e.g. for the scalar

$$f(Z) \;=\; \frac{1}{(A_\alpha Z^\alpha)\,(B_\beta Z^\beta)}, \tag{16}$$

with simple poles on the two planes $A \cdot Z = 0$ and $B \cdot Z = 0$. Restricted to a line L_x the poles are two points; the contour separates them; and the residue gives a rational solution of the wave equation, regular wherever the pole planes miss L_x. Positive frequency now acquires geometric content: the field is positive-frequency precisely when f extends holomorphically over all of $\mathbb{PT}^+$ — for (16), when both pole planes can be held inside the closure of $\mathbb{PT}^-$ — which is the correspondence recorded in (11). *That* the division by s carries the frequency splitting is, in the twistor literature, a fundamental and cherished fact; sourcing it in real causal data is the business of Part V.

The helicity ladder, and its famous asymmetry. Reading (11) across the physical fields: $h = 0$, homogeneity -2, the wave equation; $h = \pm\frac{1}{2}$, homogeneities -3 and -1, the two chiral massless spin-$\frac{1}{2}$ equations; $h = \pm 1$, homogeneities -4 and 0, self-dual and anti-self-dual Maxwell theory; $h = \pm 2$, homogeneities -6 and $+2$, the two handednesses of linearized gravity. The visible asymmetry between a helicity and its mirror is harmless at the linear level — both sides of the ladder are covered by the transform — but it is the seed of the program's central nonlinear hurdle, the googly problem, which this paper meets at its own boundary in §44 and objection O-P4-1.

From contours to cohomology. The contour description carries a redundancy: f and $f+g_+-g_-$, with $g_\pm$ holomorphic on the two sides of the contour's neighborhood, give the same field. Functions on an overlap, modulo functions extending over the patches, is exactly a first Čech cohomology class — and that is how (11) earns its H^1: a massless field is not a holomorphic *function* on half of twistor space (of negative homogeneity there are none to be had) but an *obstruction* class, and an obstruction is precisely what a contour can detect. Course B, Lectures B.5–B.6, rebuilds this passage slowly, with the two-patch covering worked in full.

Physics intuition. A twistor is, roughly, a light ray with a size and a handedness; a spacetime point is the family of rays through it. The transform says a massless field — an object defined by a differential equation at every point — is equally an object defined by *regularity in one half* of ray space, with no equation left. The equation has been traded for holomorphy. Which half carries the field is, as (11) records and Part V exploits, precisely its frequency sign.

Close of Part II, in plain terms. The toolbox is now open: what a complex structure is (a quarter-turn that squares to -1, compatible with the classical structure, tamed by positivity); how one is built from a generator (the polar phase — keep the direction, divide out the speed); which self-pairings a circle allows (winding arithmetic); and what positive frequency really means (one-sided regularity in imaginary time, with twistor space waiting at full conformal symmetry). Nothing beyond linear algebra and classical analysis was needed, and nothing more will be. The next Part supplies the physics that selects among these tools: the axioms, the forced six-dimensional arena, and the two globally shared structures from which everything in this paper is derived.

Part III
The GG-Theory Arena

The axioms, the forced six-dimensional geometry, and the two globally shared real structures

This Part makes the linkage to GG-Theory exact: which constitutional clauses do the selecting, which theorem supplies the geometry, and how a charged field becomes a pair of real harmonics on a real circle. All results are inherited in completed tense; each is restated here with enough precision to carry the proofs of Parts IV–V.

15 Part III in plain English: the arena, and the two structures it supplies

This paper does not build the framework it uses. Three constitutional clauses, the theorem that forces the six-dimensional geometry, an integer ledger, a boundary law and an actualization layer are all inherited from companion papers. This Part restates each at the precision the later proofs need, and names the paper that owns it.

The output is short. Two real structures, both supplied by the arena and both attached to every system in the universe: a compact, non-spatial phase circle, and the constitutional clock together with a positivity clause. Everything this preprint derives comes from those two and from nothing else, and Table 2 records which result draws on which, alongside the classical mathematics imported to reach it.

Restating an inherited result is not the same as proving it, and the table is there so that the difference stays visible. Where a clause is doing real work in a later theorem, a reader should be able to go to the paper that owns the clause and check that it says what is claimed here.

16 The three constitutional clauses this paper uses

The GG-Theory constitution (Arisaka, 2026, A1-Constitution) rests on four axioms; this preprint uses three. **AX-1** (locality–causality) requires all registered physics to live on a Lorentzian causal manifold, read through retarded, microcausal protocols: it supplies a global *time orientation* — the causal order — and, applied to interacting charged matter, forces the two structures geometrized below. **AX-2** (geometry–gauge equivalence) requires every structure the dynamics cannot avoid to be realized as geometry: one composite connection on a principal bundle, with all physical content in curvature and holonomy invariants. AX-2 is verifiably complex-free: its connection is a real one-form, its holonomies real angles. **AX-3** (the operational dictionary) requires a bulk theory to specify how boundary observers register outcomes — instruments, channels, and, decisively for this paper, an *operational positivity* clause. AX-1 and AX-3(iii) are the two selection principles of

Part IV: the first orients the clock, the second picks the sign of its polar phase.

Because everything in this paper hangs on them, the three load-bearing clauses deserve more than the summary above; the following renders each at the resolution of Arisaka, 2026, A1-Constitution, with the fourth axiom added for completeness.

AX-1 in full. Locality–causality is not one requirement but a bundle: interactions built from products of fields at coincident points; retarded support — no response before its cause; and microcausality — spacelike-separated observables commute. Two consequences are forced rather than assumed, and both matter here. An interacting local theory is singular at coincident points and so cannot be specified without a resolution scale and the flow connecting resolutions — the renormalization group, made compulsory. And local, causal interaction of *charged* matter cannot proceed by action at a distance: a phase frame must be carried at every point with a connection compensating its arbitrariness — the gauge principle, read as a consequence. A scale and a phase: AX-1 hands exactly two structures to the next axiom.

AX-2 in full. The geometry–gauge equivalence principle has two clauses. *Locked equivalence*: along any timelike curve there is a combined choice of coordinates, frame, and gauge in which the composite connection is flattened to first order — Einstein's elevator, upgraded to geometry-plus-gauge as one locked structure. *Invariant physics*: all physical content resides in curvature and holonomy invariants of that one connection; "forces" are artifacts of unlocked descriptions. Both clauses are complex-free — a real connection, real curvature, real holonomy angles — and the operative reading is a single rule: whatever the dynamics cannot avoid must be realized as geometry. In the cohort's own words: AX-2 *builds* the fiber.

AX-3 in full. The operational axiom declares that a bulk theory is not physically defined without a fixed dictionary from bulk configurations to registered boundary physics,

$$\mathcal{D}(B) \;=\; \big(\mathcal{H},\ \rho,\ \{\mathcal{I}_\alpha\},\ \{\Phi_{t,s}\},\ \ldots\big), \tag{17}$$

with ρ a boundary state, $\{\mathcal{I}_\alpha\}$ measurement *instruments* in Kraus form with POVM semantics, and $\Phi_{t,s}$ completely positive trace-preserving evolution maps, the whole subject to three compatibility clauses: retardedness (AX-1), gauge–geometry covariance (AX-2), and *operational positivity* — clause (iii), the one this paper leans on at T-P4-2. Measurement is thereby part of the theory's syntax, not a metaphysical addendum; and the instruments' membership in the dictionary is why observer-adapted structures (which modes a detector counts; which infrared background it calls vacuum; which horizon boundary condition it declares) are boundary data throughout Part VI. AX-3 *collapses* the fiber that AX-2 built.

AX-4, for completeness. Maximum symmetry, minimum freedom: no continuous structure is admissible unless a prior clause compels it. Its role here is quiet but real — it closes Theorem 0's count from above (no seventh dimension) and forbids the unforced extras that would otherwise let a phase float free of the one circle.

Physics intuition. The three axioms are a relay. AX-1 says: to be local, causal, and charged, you must carry a zoom knob and a dial. AX-2 says: if you must carry them, they are places — geometry, not bookkeeping. AX-3 says: and here is what a four-dimensional being can actually read of them — states, channels, instruments, with positivity. The first hands two burdens to the second; the second makes them real; the third makes them readable. The complex numbers of this paper are born in the gap between the second and the third: real geometry, read by a bounded reader.

17 Theorem 0: the forced arena

Theorem 0 of Arisaka, 2026, A2-GG6 establishes that AX-1 and AX-2 together force the arena

$$\mathcal{M}_6 \;=\; \mathcal{M}_4 \times I_r \times S^1_\theta, \qquad ds_6^2 \;=\; e^{-2r}\,\eta_{\mu\nu}\,dx^\mu dx^\nu \;+\; dr^2 \;+\; d\theta^2, \tag{18}$$

with I_r a non-compact *scale* direction (the geometrized renormalization group) and S^1_θ a compact *phase* circle (the geometrized gauge fiber). Three properties carry everything below. *Non-spatiality:* the two extra directions are classified by their generators — a dilatation and a compact charge, not translations — so no signal propagates along them and no boundary protocol resolves a position on them; the bulk isometry algebra is $\mathfrak{iso}(1,3) \oplus \mathbb{R}_D \oplus \mathfrak{u}(1)_Q$. *Global staticity:* η is Minkowski and the warp depends only on r, so ∂_t is a global hypersurface-orthogonal timelike Killing field — the *constitutional clock* — and the uniqueness clause of Lemma 11.1 applies globally. *Reality:* every datum in (18) — metric, interval, angle, connection — is real.

The proof of Theorem 0 is a six-step forcing chain worth possessing in outline, because each step is load-bearing later. *Step 1*: AX-1 is defined relative to a light cone, so the registered arena is Lorentzian. *Step 2*: locality plus interaction forces the renormalization scale (AX-1's first burden). *Step 3*: AX-2 promotes it to a coordinate — the scale direction I_r, along which the connection component is the running coupling and the β-function is a covariant derivative. *Step 4*: locality plus charge forces the periodic gauge phase (AX-1's second burden). *Step 5*: AX-2 promotes it to the compact direction S^1_θ, whose Wilson-line holonomy is the gauge-invariant datum. *Step 6*: the count closes two-sidedly — from below because deleting either direction breaks AX-1 itself (no scale: ill-defined at short distance; no phase: action at a distance), from above because AX-4 admits no third structure. Two further properties are proved en route and used constantly here: the added directions carry *no light cone* (their generators are a dilatation and a compact charge, not translations — non-spatiality by classification, not by decree), and motion along θ is a gauge transformation, hence not a physical displacement. That last clause is the deep difference from every spatial-circle construction: a Kaluza–Klein circle is somewhere one could in principle look; the phase circle is not a place at all, which is why its unreadability — and everything superselection-shaped that follows from it — is constitutional rather than practical.

Physics intuition. Do not picture six axes. Picture the stage (spacetime), a zoom knob attached to it (the scale), and a dial at every point (the phase). Theorem 0 says: charged, causal physics cannot be written without the knob and the dial, and the constitution obliges both to be geometry. The knob is not a corridor; the dial is not a room. That is the whole of "non-spatial," and it is why the dial's angle can be physical only through differences — holonomies — which is where the complex exponential will come from.

18 The integer ledger: what BI-6 fixes before this paper begins

One more inheritance is load-bearing and deserves its own section: the five-integer ledger $(M, N, L;\ J, K) = (1, 3, 5;\ 7, 35)$, introduced with the arena and proved forced — not fitted — by the anomaly-closure architecture of Arisaka, 2026, A3-BI6. Each integer carries a primary structural meaning: $M = 1$, the tensor channel is primitive and unique; $L = 5$, the unified simple block is the defining five-dimensional one; $N = 3$, the baryonic trace ratio isolates a triplicity inside that block (whose later echoes are color and the three generations); $J = 7$, the centered odd closure built from the triplicity; and $K = 35$, the rectangular and quadratic lock index, tied by the closure identities

$$K \;=\; L \cdot J \;=\; M^2 + N^2 + L^2 \;=\; 35. \tag{19}$$

Two entries of the ledger do direct work in this paper. $M = 1$ is the winding unit: the statement that charge quantization comes in one indivisible tensor seam is what makes the circle's characters *integers* and superselection *discrete* (T-P4-1, Lecture C.6). And $L = 5$ is the angle alphabet: with one circle and one five-block — and, decisively, no second flavor torus on which an independent angle could float — every first-order physical phase is assembled from steps of $\pi/5$, which is why the CP-violating loop holonomies of T-P4-5 are lattice-locked and why $2\pi/5$ appears where a free parameter would otherwise sit. The ledger is the reason this paper may say "quantized" and "locked" where other frameworks must say "measured."

Physics intuition. A dial that can be re-zeroed freely carries no numbers of its own. What BI-6 adds is a tick-mark structure: the anomaly bookkeeping closes only for one integer packet, and those integers become the dial's engravings — how many windings make a charge (M), how many steps make the flavor pentagon (L). The circle supplies the geometry of the phase; the ledger supplies its arithmetic.

19 The fiber dictionary: charged fields as real harmonic pairs

Single-valuedness on the compact circle gives every bulk matter field a real harmonic decomposition; the charge-n sector is the pair of real weight-n harmonics

$$\Psi_n(x;\theta) \;=\; a_n(x)\cos n\theta \;+\; b_n(x)\sin n\theta, \qquad R_\alpha : (a_n, b_n) \mapsto R(n\alpha)(a_n, b_n), \tag{20}$$

and the textbook complex charged field is the packaging $\psi_n = a_n + ib_n$ of Proposition 10.3. Under this dictionary: charge is the winding number n; charge quantization is compactness ($n \in \mathbb{Z}$); a gauge transformation is a fiber shift $\theta \mapsto \theta + \alpha(x)$; the covariant derivative is the horizontal lift, acting on the n-harmonic as $\partial_\mu - inA_\mu$; the Aharonov–Bohm phase is a real holonomy angle of the real connection; and complex conjugation is orientation reversal $\theta \mapsto -\theta$. The photon is the connection itself — not a section of any associated bundle — and therefore correctly carries no dial.

20 The boundary law: OGGEP and GDOS

The dictionary demanded by AX-3 is not left abstract in the cohort; Arisaka, 2026, A5-GDOS constructs it, and four of its structural results frame everything this paper does with observers, horizons, and the infrared. *First, the projection is forced.* A boundary protocol reads spacetime records sorted by a light cone; Theorem 0 gave the fiber directions no light cone; hence no operation resolves where a system sits along I_r or around S^1_θ, and every act of registration marginalizes the fiber. Openness is constitutional; unitary closure is the unreachable idealization in which the fiber is never coarse-grained. *Second, the projection is always on.* It happens at every spacetime point, observer or no observer; "measurement" is its macroscopic record-forming special case and decoherence its microscopic one. *Third, the boundary law has a derived form.* Exact projection gives a memory-kernel equation; complete positivity is inherited from unitary conjugation plus partial trace; and in the Markov regime the generator takes the Gorini–Kossakowski–Sudarshan–Lindblad form — from which the textbook quantum postulates follow as theorems (Hilbert space by GNS, Schrödinger as the closed limit, Born form by Gleason, Lüders by conditioning, decoherence, einselection, Bell). *Fourth, and decisive for Parts IV–VI: the two fibers carry the two halves of quantum mechanics.* The compact phase circle carries the coherent, unitary, interference-generating half — transport along it is pure phase; while the non-compact scale direction carries the open, irreversible half — its forward renormalization-group semigroup is the dissipative generator, and decoherence is the loss of θ-coherence induced by coarse-graining along r. This paper's subject matter is the first fiber's half; but its boundaries — which vacuum an instrument declares, which soft cloud it stands in, what a horizon traces out — are all items of the second half's dictionary, which is why the infrared and horizon discussions of Part VI could be stated as *port* assignments rather than as ambiguities of the bulk.

Physics intuition. The bulk is a six-dimensional book; the boundary observer reads a four-dimensional shadow of it, always, whether or not anyone calls the reading an experiment. What the shadow cannot show — the dial angle, the zoom setting — comes out as probability and irreversibility; what it can show — differences, rates, records — comes out real. Quantum mechanics is the grammar of reading a book through its shadow, and this paper's claim is that the grammar's one "imaginary" rule is the shadow's compression of the dial and the clock.

21 The actualization layer: GRIP, and why records are real

One more story of the parent program bears directly on this paper's arc, because it is where outcomes — the *real* end of the real-to-complex-to-real chain — are fixed. Arisaka, 2026, A6-GRIP builds the runtime in four pillars: the total geometric–gauge object (the one connection, as a dynamical whole); the covariant conservation identity (the four boundary Ward currents — electric, baryonic, informational, metric — that any lawful runtime must preserve); the geometric free-energy functional (a strict Morse landscape ranking admissible boundary configurations); and GRIP, the gauge-runtime invariance pipeline: descent proceeds in a finite, ordered cascade of Morse steps, each transition costing a quantized action and the whole bounded by the BI-6 budget, halting at the ground geometric attractor. Carried into quantum mechanics by Arisaka, 2026, A8-QM, this supplies the measurement story this paper inherits silently every time it says "registered": the pointer basis is forced by bulk locality; the single outcome is the deterministic capture of the joint system-and-apparatus configuration by one attractor basin, the registered datum being a *contextual joint holonomy* — an element of the circle, a real angle class; and the Born weights' square-modulus form is Gleason's (Gleason, 1957), with the run-by-run value set by bulk boundary conditions inaccessible to the four-dimensional reader. (What remains open there — the proof that basin-capture statistics reproduce the Born measure — is the parent program's named conjecture, not this paper's burden.) Two consequences matter here. Every registered output of physics is a real quantity — an angle class, a count, a rate — so the complex layer of the formalism begins *after* preparation and ends *before* registration: it is genuinely middleware. And the coupling textures whose loop holonomies measure CP chirality (T-P4-5) are not free functions but GRIP outputs — stabilized attractor data — which is why the parent program can derive their angles and this paper can call them lattice-locked.

Physics intuition. If GDOS says how the shadow is read, GRIP says how the ink dries: a measurement completes when the joint configuration rolls into a basin it cannot leave, and what is written is an angle — a dial reading, real as any pointer. Between the real bulk that supplies the story and the real record that ends it, the complex amplitude lives only in the telling.

22 The arc, assembled: real bulk, complex middleware, real records

Part III can now be summarized as a single pipeline, which is the frame the rest of the paper fills in. The *inputs* are real: an arena (Theorem 0), one connection (AX-2), an integer ledger (BI-6), matter as real harmonic pairs on the circle (§19). The *outputs* are real: GDOS probabilities and rates, GRIP records, holonomy classes. In between sits the layer this paper is about: amplitudes, phases, state spaces — the compressed notation a fiber-blind, tube-blind reader is forced into. Parts IV and V prove the compression theorems (which real structures the notation compresses, and why the compression is unique); Part VI patrols its boundaries. Nothing complex enters the pipeline's ends; the question of this paper is only, and exactly, the middle.

The pipeline on the two-fiber anatomy. The cohort's recent campaign (GG-W1-Symmetry; Arisaka, 2026, P5-CPT; Arisaka, 2026, A8-QM) states the shape of this pipeline as one exact sentence: *holonomy requires closed loops, and an interval has none.* The compact circle S^1_θ is nothing but closed loops — everything on it is holonomy, quantized and protected, its transport a group; the scale direction is an interval — no loops, no holonomy, transport composing only forward, a semigroup. The pipeline's anatomy follows. The *middle* is complex because the middle is where circle transport is being described: weight decomposition under a circle action is what manufactures the i (Part IV), and the positivity clause selects its orientation — the i is the circle's, on loan to the notation. The *ends* are real because the ends are interval events: preparation and registration are one-way writes along the fiber that carries no phase — so a record is real because of the fiber it is written *along*, and permanent because its content is deposited in quantities the *other* fiber protects.

The custody principle of §39 is this anatomy in legal language — the dictionary owns the sectors; the geometry owns the i. And the paper's two title questions each get a one-clause topological answer: why complex numbers? Because one fiber is a circle. Why real records? Because the other is an interval.

23 The two globally shared structures, named

Everything this preprint derives comes from exactly two real structures, both supplied by the arena and both attached to *every* system in the universe:

$$\textbf{G1:}\ \text{the phase circle } S^1_\theta \ \ (\text{generator } \partial_\theta), \tag{21}$$

$$\textbf{G2:}\ \text{the constitutional clock } \partial_t \ \ \text{with the positivity clause of AX-3.} \tag{22}$$

G1 will supply the gauge phase and everything quantized about it; G2 will supply the state-space complex structure and everything holomorphic about it; and their consistency on charged sectors will be the particle–antiparticle pairing. The inheritance is summarized in Table 2.

Table 2: What each result of this preprint uses, and where it lives in the parent program. One row per numbered result. The middle column names the GG-6 axiom, theorem or ledger entry the result stands on; the right column names the classical mathematics imported to reach it, with an em dash where none is needed. The table is here so that a reader can see at a glance which parts of the argument are new to this paper and which are inherited, and can go to the owning paper for the second kind.

Result (this paper)	**GG source**	**Imported classical mathematics**
T-P4-1 (circle source)	AX-2; Theorem 0 (G1)	harmonic analysis on S^1
T-P4-2 (clock source)	AX-1; AX-3(iii) (G2)	Ashtekar and Magnon (1975); Kay (1978)
T-P4-3 (antiparticles)	G1 + G2 jointly	—
T-P4-4 (one shared i)	G1, G2 global	Renou *et al.* (2021) (as constraint)
T-P4-5 (discrete table)	orientations of G1, G2	—
T-P4-6 (Hardy level)	G2	Paley and Wiener (1934)
tube level	G2	Streater and Wightman (1964)
T-P4-7 (twistor level)	G2, conformal	Penrose and MacCallum (1973); Eastwood *et al.* (1981)
T-P4-8 (interacting)	G1, G2 asymptotic	Haag–Ruelle (Haag, 1992)
T-P4-9 (infrared door)	AX-3 (ports)	Fröhlich (1973); Pizzo (2003)
lattice-locked phases	BI-6 ledger ($L = 5$)	—
winding integrality	BI-6 ledger ($M = 1$)	—
textures; real records	GRIP (Arisaka, 2026, A6-GRIP)	Morse theory

Physics intuition. The whole construction can be said in one breath: nature carries one dial and one clock, the same pair everywhere. The dial cannot be read, so only its differences (holonomies) are physics; the clock cannot be run backward, so only one frequency orientation is physics. Complex numbers are what a four-dimensional observer writes down when forced to log differences of an invisible dial and orientations of an irreversible clock in a single symbol.

Close of Part III, in plain terms. The stage is set and it is entirely real: an arena forced by locality and equivalence (a stage, a zoom knob, a dial at every point); an integer ledger engraving the dial; a boundary law that reads the bulk through an always-on projection; and an actualization layer that writes real records. Two structures on this stage are shared by everything in the universe — the phase circle and the causal clock — and the paper's whole claim is that they are where the complex numbers come from. The next Part proves it: five theorems, from the circle's quarter-turn to the discrete-symmetry table, with the antiparticle emerging as the handshake between the two sources.

Part IV
The Two Real Sources

Five theorems: the circle, the clock, the antiparticle handshake, the shared i, and the discrete-symmetry table

24 Part IV in plain English: the circle and the clock

This is where the derivation starts. Five theorems, best read as two sources and three consequences.

The circle supplies the gauge phase. It is a compact real dimension of the bulk, a charged field is a pair of real harmonics on it, and the complex structure of charged matter is the rotation the circle already carries. The clock supplies the state-space i: causal time orientation together with energy positivity select the polar phase of the time-translation generator, uniquely, on a static arena. The staticity is a condition and not a decoration — its gravitating generalization is inherited rather than owned, and Part VIII says so again. Neither derivation uses the other.

Then the three consequences. That the two sources agree on charged sectors is the particle–antiparticle pairing, and it is the first real check in the paper, because two independent derivations could have disagreed and did not. That the i is one i shared by everything, rather than one per system, is what a composite system requires and what the 2021 experiment tests. And the discrete symmetries follow from the two orientations as a single table rather than as five separate arguments.

25 The circle: the gauge phase as real geometry

Theorem 25.1 (**T-P4-1**: the circle source)**.** *On each weight-$n \neq 0$ sector* (20)*: (i) the charge-normalized circle generator $J_Q = \partial_\theta / Q$ (with $Q = \pm n$ the signed charge) is a complex structure in the sense of Definition 10.2, commuting with the covariant dynamics and with all gauge transformations; (ii) $(V_n, J_Q) \cong$ the standard complex charged field of charge n, with complex conjugation realized as circle-orientation reversal; (iii) J_Q is unique up to sign: the commutant of the irreducible weight-n circle action contains exactly the two quarter-turns; (iv) on the weight-zero sector the construction is empty — ∂_θ acts as zero — and by Lemma 12.1 the only invariant self-pairing at $n \neq 0$ is the charge-conserving $\psi\bar{\psi}$.*

Proof. (i) On the pair (a_n, b_n), ∂_θ acts as $(a, b) \mapsto (nb, -na)$, so $J_Q^2 = -\mathbf{1}$ on either orientation; the symplectic form (1) is built from θ-independent data, so the isometry preserves it; the principal connection is θ-invariant, so $[D_\mu, \partial_\theta] = 0$; a gauge transformation is a fiber shift and commutes with ∂_θ. (ii) is Proposition 10.3 applied fiberwise, plus the horizontal-lift computation $D_\mu \psi_n = (\partial_\mu - inA_\mu)\psi_n$; under $\theta \mapsto -\theta$, $(a, b) \mapsto (a, -b)$, i.e. $\psi \mapsto \bar{\psi}$. (iii) The commutant of an irreducible $SO(2)$ action on a plane is spanned by rotations; its solutions of $J^2 = -\mathbf{1}$ are the two quarter-turns. (iv) is Lemma 12.1. ■

Physics intuition. Everything "complex" about a charged field is dial-reading. The field strength of its phase is how fast the dial winds across spacetime (de Broglie); the Aharonov–Bohm shift is how much the dial turns around a loop; charge is how many times the dial turns per turn of the fiber. None of these requires i — they require an angle, which is what the sixth dimension is.

26 The clock: the state-space i

Theorem 26.1 (**T-P4-2**: the clock source)**.** *On the real solution space of any free (or quasi-free) bulk field on the arena* (18)*, gapped case: the polar phase J_A of the time-translation generator (Lemma 11.1) is a complex structure, unique among dynamics-commuting positive-energy structures (Ashtekar and Magnon, 1975; Kay, 1978); it is the Hilbert transform in constitutional time; and the selection uses only real constitutional data: AX-1 orients the clock, AX-3(iii) selects the sign, Theorem 0's global staticity makes the choice canonical. The same holds for Majorana fermions on their Euclidean solution space, and for the exact ground state of any gapped quadratic system, whose Gaussian covariance Γ satisfies $J = 2\Gamma\sigma = J_A$.*

Proof. Existence and the listed properties are Lemma 11.1; uniqueness on a stationary region is the cited selection theorem, and Theorem 0 supplies global stationarity. The Hilbert-transform identification: on each frequency-ω plane J_A is the quarter-period advance, which is (7) frequency by frequency. The quadratic-ground-state identity follows from symplectic diagonalization: with stiffness K, the ground covariance is $\Gamma = \frac{1}{2}\,\mathrm{diag}(K^{-1/2}, K^{1/2})$ in (φ, π) blocks, and $2\Gamma\sigma$ reproduces (5) for $A = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ -K & 0 \end{smallmatrix}\right)$ by inspection. ■

Physics intuition. Ask what a neutral field's phase *is*, given that it has no dial: it is the field's position in its own cycle — where each mode stands between its cosine and its sine. "Multiply by i" means "advance every mode a quarter of its own period." That operation is real (a principal-value integral over the history), non-local in time (you must know the whole cycle to know a quarter of it), and unique once you insist that energy be positive — which is the constitution's own clause, not a taste.

27 Antiparticles: the handshake

Theorem 27.1 (**T-P4-3**: circle–clock consistency)**.** *On the on-shell charge-n sector, the two structures agree, $J_A = J_Q$, precisely under the Feynman–Stückelberg pairing — charge $+n$ with positive frequency, charge $-n$ with positive frequency — and anti-agree on the opposite pairing. Explicitly, on the real mode basis: writing the total dial angle $n\theta - Et$ $(E > 0)$,*

$$J_Q:\ \cos(n\theta - Et) \mapsto -\sin(n\theta - Et) \quad \textit{and} \quad J_A:\ \cos(n\theta - Et) \mapsto -\sin(n\theta - Et), \tag{23}$$

while on the wrong pairing $\cos(n\theta + Et)$ *the two maps differ by a sign. Hence the sign ambiguity of T-P4-1(iii) is resolved by T-P4-2's positivity, there is* one *global complex structure* J *— sourced by the clock, geometrized by the circle wherever charge lives — and the existence of antiparticles is the consistency condition between the sixth dimension and causal time.*

Proof. J_Q is the quarter-turn $\theta \mapsto \theta + \pi/2n$ on the $+n$ orientation, giving $\cos(n\theta - Et + \pi/2) = -\sin(n\theta - Et)$; J_A is the quarter-period advance $t \mapsto t - \pi/2E$, giving $\cos(n\theta - Et + \pi/2)$ likewise. On $\cos(n\theta + Et)$ the same two substitutions produce opposite signs. Re-labeling the negative-frequency, charge-n modes as positive-frequency, charge $-n$ modes (under which $J_Q = \partial_\theta/Q$ flips its normalization with Q) restores agreement on every physical mode; linearity extends the identity from modes to the space. ■

Physics intuition. The circle knows the plane of rotation but not the direction; the clock knows the direction. Nature must use both, so their orientations must mesh — and the unique meshing rule is the one every textbook imposes by hand: pair opposite charge with opposite frequency. Antiparticles are not an extra ingredient of relativistic quantum theory; they are the gear alignment between its two real sources.

28 One shared i, and composite systems

Theorem 28.1 (**T-P4-4**: the shared complex structure)**.** *(i) The global* J *of T-P4-3 is common to all sectors and all subsystems, because its sources G1, G2 are globally shared. (ii) For subsystems carrying the same* J*, the complex tensor product is the quotient of the real one by the shared-*J *identification,*

$$V \otimes_{\mathbb{C}} W \;\cong\; (V \otimes_{\mathbb{R}} W)\,/\,\mathrm{span}\{\, Jv \otimes w \;-\; v \otimes Jw \,\}, \tag{24}$$

with the induced complex structure well defined exactly because J *is shared; the Fock construction extends* J *with no new complex input, and entangled states are elements of the real quotient. (iii) The experimental exclusion of real quantum mechanics (Renou* et al.*, 2021) constrains precisely the* unshared *case — real tensor products of separately-real state spaces — and is satisfied here by construction.*

Proof. (ii) The quotient identifies the two real actions of J; the induced operator $J \otimes \mathbf{1} \equiv \mathbf{1} \otimes J$ squares to $-\mathbf{1}$ and is well defined iff the identified actions agree — i.e. iff $J_V = J_W$ under the identification; for unshared structures no canonical complex quotient exists, which is the abstract content of the network-Bell pathology the experiment exploits. (i) and (iii) are then immediate. ■

29 The discrete-symmetry table, derived

Theorem 29.1 (**T-P4-5**: orientation bookkeeping)**.** *Let* C, P, T *act as orientation reversals of circle, space, and clock respectively. Then:* C *is* J*-linear (it reverses* ∂_θ *and the signed charge together, leaving* $J_Q = \partial_\theta/Q$ *invariant);* T *is* J*-antilinear (it reverses the polar phase of the generator);* P *is* J*-linear; hence a discrete symmetry is antiunitary if and only if it contains* T*. The total reversal*

CPT preserves each Feynman–Stückelberg-paired mode set and is a symmetry of the construction for every coupling texture built from real data; CP alone is a symmetry iff the texture is achiral on the circle — every closed-loop (rephasing-invariant) angle of the texture lies in $\{0, \pi\}$. In the parent program the loop angles are lattice-locked, the physical quark CP phase being two pentagonal steps, $\delta = 2\pi/5$ (Arisaka, 2026, A9-SM).

Proof. Under $\theta \mapsto -\theta$ the weight-n sector maps to weight $-n$ and $\partial_\theta \mapsto -\partial_\theta$; since J_Q is normalized by the signed charge, $J_Q \mapsto (-\partial_\theta)/(-Q) = J_Q$. Under $t \mapsto -t$ the frequency orientation flips, so $TJ_AT^{-1} = -J_A$. (*anti*)linearity multiplies under composition, giving the table. CPT maps $\cos(n\theta - Et) \mapsto \cos(n\theta - Et)$ on each paired mode; real texture data (moduli, winding sums, metric contractions) are even under total reversal, and reversing the sign of every texture angle while conjugating every loop is a relabeling. A circle reversal alone negates every loop angle; invariance up to re-zeroing forces each loop angle to equal its own negative mod 2π. ∎

Physics intuition. Why is time reversal the odd one out — antiunitary, where C and P are unitary? Because of the three orientations a discrete symmetry can flip, only the clock's is the one i is made of. Flip space, and nothing happens to i; flip the dial, and the charge label flips with it, so i survives; flip the clock, and "a quarter period ahead" becomes "a quarter period behind" — i becomes $-i$, which is antilinearity. The textbook table is a map of which reversals touch the sources of i.

Close of Part IV, in plain terms. The two sources have now done their work. The circle gave charged matter its phase, its quantized charge, and its superselection; the clock gave every field — charged or neutral — its state-space i, unique once energy is positive; their agreement on charged sectors forced the existence of antiparticles; their shared global character gave composite systems one i and entanglement a real presentation; and their orientations, flipped one at a time, generated the C, P, T table that textbooks assign by hand. What remains of the original question is its deepest half: Penrose's holomorphy. The next Part crosses that bridge — from the Hilbert transform, through the forward tube, to twistor space itself.

Part V
The Bridge: Positive Frequency is Holomorphy

Hardy space, the forward tube, and the twistor correspondence as one structure at three symmetries

30 Part V in plain English: the bridge, which is what the title claims

The title of the paper is this Part. Positive frequency is holomorphy: at the Hardy level for a single mode, at the tube level for a field, and at the twistor level once conformal symmetry is available. The claim is that these are one structure seen at three symmetries, not three analogies that happen to rhyme.

Every component is established mathematics and the deepest two belong to the twistor school. What is new here is the direction of explanation. In the twistor reading, holomorphy is primitive and the frequency splitting is one of its shadows; in the reading assembled here, the real causal-positive data come first and holomorphy is what those data look like at sufficient symmetry. The twistor construction loses nothing by this. It gains its motivation.

Six worked examples close the Part, from the analytic signal to a CKM plaquette computed with no complex numbers anywhere in it. They are here because a claim of this shape is easy to state and easy to believe without checking, and an example a reader can work by hand is the cheapest way to find out whether it is true.

31 The Hardy level

Theorem 31.1 (**T-P4-6**: the derived J generates holomorphy)**.** *The complex structure selected in T-P4-2 is the generator of the Hardy splitting: for real finite-energy histories,*

$$f \text{ positive-frequency} \iff f = \text{boundary value of } F \in H^2(\operatorname{Im} z > 0) \iff \mathsf{H} f_+ = -\,i\, f_+, \tag{25}$$

where $f_\pm = \frac{1}{2}(\mathbf{1} \pm i\mathsf{H})f$ are the frequency halves and $F(t+is) = \int_0^\infty \hat{f}_+(\omega)\, e^{i\omega t - \omega s}\, d\omega$ is the explicit extension, which satisfies the Cauchy–Riemann equation $\partial_s F = i\,\partial_t F$ for $s > 0$ and recovers $f = 2\operatorname{Re} F$ on the boundary; a negative-frequency component instead grows as $e^{+|\omega|s}$. Consequently the state-space i of GG-Theory is, verbatim, the infinitesimal form of upper-half-plane holomorphy in complexified constitutional time.

Proof. The equivalences are Theorem 13.3 together with the Fourier multiplier form (7): on frequency support $[0, \infty)$, H acts as $-i$; the displayed extension converges for $s > 0$ precisely because the support is one-sided, and term-by-term differentiation gives $\partial_s F = i\,\partial_t F$; for support on $(-\infty, 0]$ the same formula produces $e^{+|\omega|s}$. The identification with T-P4-2's J is Lemma 11.1's time-domain form. ■

Physics intuition. The analytic signal of the engineer is the whole theorem in miniature. Given a real voltage trace, form $f + i\mathsf{H}f$: a complex "signal" whose modulus is the envelope and whose argument is the instantaneous phase. Engineers call this a trick. The theorem says it is not a trick: histories with one frequency orientation *are* the boundary shadows of functions living regularly in one half of complexified time, and the "i" being attached is the same derived J that quantum mechanics uses. The state space of physics is an engineering trick that turned out to be geography.

32 The tube level

The field-theoretic form replaces the single time variable by all of them. Positive-frequency solutions of a free field extend holomorphically to the *forward tube*

$$\mathcal{T}^+ \;=\; \{\, x + iy \;:\; y \in V^+ \,\} \qquad (V^+ = \text{open future timelike cone}), \tag{26}$$

and in any Wightman theory the vacuum correlation functions are boundary values of functions holomorphic in (permuted extensions of) $\mathcal{T}^+$ (Streater and Wightman, 1964). Tube analyticity is the standing field-theoretic expression of causal positivity: the tube is the complexified constitutional clock, one imaginary direction per timelike one, and holomorphy in it says exactly that the theory's frequencies carry one orientation with respect to every causal observer at once. For a massive theory, this is where analyticity stops.

Physics intuition. Why a *tube*? Because "positive frequency" must mean the same thing for every inertial clock, and each clock contributes its own imaginary direction in which histories of the right orientation decay. The tube is the intersection of all those one-sided regularities — the geometric statement that the universe's frequency orientation is Lorentz-invariant.

33 The twistor level

Theorem 33.1 (**T-P4-7**: the twistor division is the frequency splitting)**.** *For free massless fields of helicity h: (i) the Penrose transform* (11) *identifies positive-frequency solutions with $H^1(\mathbb{PT}^+, \mathcal{O}(-2h-2))$ and negative-frequency solutions with the corresponding classes on $\mathbb{PT}^-$; (ii) along any timelike worldline, the pullback of a positive-frequency field is a Hardy-class function of proper time, so the twistor division restricts on every clock to the Hardy splitting of T-P4-6; (iii) hence the division of twistor space by the sign of the helicity form* (9) is *the causal-positive frequency splitting generated by the derived J, conformally packaged: a massive field's positivity buys analyticity in one complexified direction at a time, a conformally invariant massless field's positivity buys the whole tube at once, and twistor space is the geometry that linearizes exactly that enhancement.*

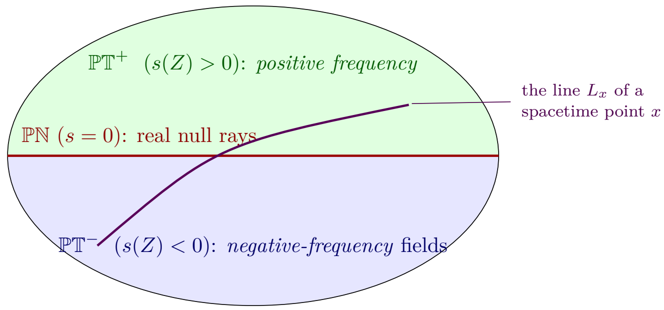


Figure 3: Projective twistor space divided by the sign of the helicity form (9). The five-real-dimensional null hypersurface $\mathbb{PN}$ carries the real null geodesics; a spacetime point x appears as a projective line L_x; and positive-frequency massless fields are exactly the cohomology carried by the upper half, $H^1(\mathbb{PT}^+, \mathcal{O}(-2h-2))$. Theorem T-P4-7 identifies this division with the causal-positive frequency splitting generated by the derived J.

Proof. (i) is Theorem 14.2, imported with its original attribution (Eastwood *et al.*, 1981; Penrose and MacCallum, 1973). (ii) A positive-frequency massless field has momentum support on the future null cone; its restriction to a timelike worldline with proper time τ has frequency support in $[0, \infty)$ (the Minkowski pairing of a future-null momentum with a future-timelike velocity is positive), hence is Hardy by Theorem 13.3. (iii) The helicity form is the twistor-space expression of frequency sign — fields on $\mathbb{PT}^+$ are exactly those extending to the forward tube in the incidence relation's complexified x — so (i) and (ii) exhibit one splitting written at three symmetries: a single clock (Hardy), all clocks (tube), all conformal clocks (twistor). ■

Physics intuition. Take the double slit's phase dial and ask how much symmetry can be loaded onto it. With one observer, the dial's orientation gives Hardy analyticity in that observer's imaginary time. Demand agreement among all inertial observers: the tube. Demand agreement under the full conformal group — the natural symmetry of massless light — and the "imaginary time" directions knit into a compact complex three-space, and the dial's orientation becomes the division of that space into two halves. Twistor space is what the frequency dial looks like when light's full symmetry is allowed to act on it.

34 The direction of explanation

Every component above is established mathematics, and the deepest two belong to the twistor school itself. What this preprint claims — its entire claim — is the direction of explanation. In the twistor reading, holomorphy is a primitive endowment of nature and the frequency splitting one of its shadows. In the reading assembled here, the real causal-positive data come first — a compact circle, a causal order, a positivity clause, all real, all constitutional (§16) — and holomorphy is what those data *look like* at sufficient symmetry. The twistor construction loses nothing; it gains its motivation: the reason nature's massless fields organize themselves holomorphically on $\mathbb{PT}$ is that

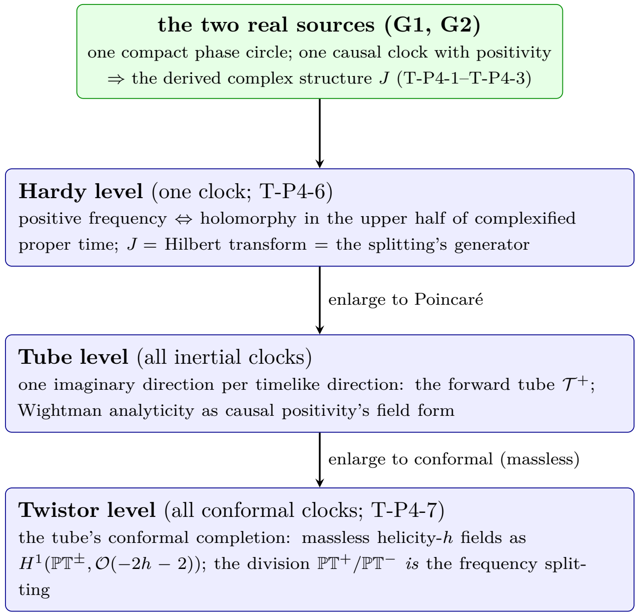


Figure 4: The holomorphy bridge. One derived complex structure, read at three symmetry levels: with a single clock, positivity is Hardy-space holomorphy; demanding agreement among all inertial clocks yields the forward tube; and for conformally invariant massless fields the tube completes to twistor space, where the frequency splitting becomes the geometric division $\mathbb{PT}^+/\mathbb{PT}^-$.

holomorphy is the conformal completion of the same real structure that makes amplitudes complex, charges quantized, and antiparticles necessary. Nothing was invented; something was found — and what was found was never the complex numbers, but the circle and the clock they compress.

35 Worked examples

35.1 Worked example I: the analytic signal, end to end

Take the elementary real history $f(t) = \cos\omega t$, $\omega > 0$. Its frequency halves are $f_\pm = \frac{1}{2}e^{\mp i\omega t}$ (conventions of (7)); the Hardy extension of the positive half is

$$F(t+is) \;=\; \tfrac{1}{2}\,e^{i\omega(t+is)} \;=\; \tfrac{1}{2}\,e^{i\omega t}\,e^{-\omega s}, \tag{27}$$

manifestly holomorphic (a function of $z = t + is$ alone), decaying in the upper half plane, with boundary value recovering $f = 2\,\mathrm{Re}\,F$; the negative half extends only downward, growing as $e^{+\omega s}$ upstairs. The Hilbert transform acts as the quarter-period advance, $\mathsf{H}f = \sin\omega t$, and one checks $\mathsf{H}^2 f = -f$: the derived J in its simplest clothing. The "analytic signal" (Gabor, 1946) $f + i\mathsf{H}f = e^{i\omega t}$ of signal processing is thus literally the boundary value of the upper-half-plane extension — T-P4-6 on one mode.

35.2 Worked example II: an elementary state through the Penrose transform

For helicity 0 the transform takes homogeneity -2: the field at x is the contour integral

$$\varphi(x) \;=\; \frac{1}{2\pi i}\oint_{\Gamma\subset L_x} f\big(ix^{AA'}\pi_{A'},\,\pi_{A'}\big)\,\pi_{E'}\,d\pi^{E'}, \tag{28}$$

over a contour on the line L_x of (8). Take the elementary state $f(Z) = \big[(A_\alpha Z^\alpha)(B_\beta Z^\beta)\big]^{-1}$ with A, B fixed dual twistors: the integrand has two simple poles on L_x, the contour separates them, and the residue evaluates to

$$\varphi(x) \;=\; \frac{1}{\rho_A(x)\,\rho_B(x)}, \tag{29}$$

a rational solution of $\Box\varphi = 0$ singular where x meets the lines of A and B ($\rho_{A,B}$ the corresponding incidence factors) — the classical computation of Penrose and MacCallum (1973); Penrose and Rindler (1986). The point for this preprint is where the *frequency* lives: the state is positive-frequency exactly when the poles can be kept on one side — when f extends holomorphically over $\mathbb{PT}^+$ — so that contour regularity in half of twistor space is, mode for mode, the Hardy condition of Worked example I. Frequency sign has become pole placement: analysis has become geometry, with nothing added.

35.3 Worked example III: a Bell state with one shared J

Write a two-mode Bell state in the real presentation. Each mode is a plane (a, b) with the shared J; the state $(|01\rangle + |10\rangle)/\sqrt{2}$ is, in the real quotient (24), the class of

$$\tfrac{1}{\sqrt{2}}(e_0\otimes e_1 + e_1\otimes e_0), \qquad J(e_i\otimes e_j) \;\equiv\; (Je_i)\otimes e_j \;\sim\; e_i\otimes(Je_j), \tag{30}$$

with e_0, e_1 real basis vectors of the two one-particle planes. The identification in (30) is exactly where the shared J earns its keep: for two *unshared* structures the two sides would be inequivalent and no complex quotient would exist — which is, abstractly, the pathology the network-Bell experiments convert into a falsification (Renou *et al.*, 2021). Entanglement itself is untouched by the real presentation: non-factorizability is a statement about the class (30), not about the number system it is written in. What the experiments exclude is unshared i; what nature runs on is one circle and one clock.

35.4 Worked example IV: horizon thermality by hand

The superselection discussion of §39 rests on one Gaussian computation, short enough to do in full. Let one field mode straddle a horizon, with the globally regular state the two-mode squeezed pairing of the exterior mode and its hidden partner: in the real presentation, a Gaussian state of two planes with covariance

$$\Gamma \;=\; \frac{1}{2}\begin{pmatrix}\cosh 2r\,\mathbf{1}_2 & \sinh 2r\,Z\\ \sinh 2r\,Z & \cosh 2r\,\mathbf{1}_2\end{pmatrix}, \qquad Z = \mathrm{diag}(1,-1). \tag{31}$$

Discard the hidden block (the GDOS partial trace across the port): the exterior covariance is $\Gamma_{\text{ext}} = \frac{1}{2}\cosh 2r\,\mathbf{1}_2 = (\bar{n}+\frac{1}{2})\mathbf{1}_2$ with $\bar{n} = \sinh^2 r$ — a *thermal* state, exactly. Now impose the horizon pairing $\tanh r_\omega = e^{-\pi\omega/\kappa}$ (κ the surface gravity): then

$$\bar{n}_\omega \;=\; \sinh^2 r_\omega \;=\; \frac{1}{e^{2\pi\omega/\kappa}-1}, \tag{32}$$

a Planck spectrum at $T_H = \kappa/2\pi$ (Hawking, 1975; Unruh, 1976), with detailed balance $\bar{n}/(\bar{n}+1) = e^{-\omega/T_H}$ holding identically. Sample numbers: for $\kappa = 0.37$ (arbitrary units), the mode $\omega = 0.2$ carries $\bar{n} = 3.47\times 10^{-2}$ and the mode $\omega = 4$ carries $\bar{n} = 3.2\times 10^{-30}$ — the exponential cliff of a genuine temperature. Two lessons: thermality is *produced* by the trace, not assumed; and the complex structure never appears — the whole computation is real covariance algebra, with J (by Lemma 38.1's logic) untouched throughout.

35.5 Worked example V: the CKM plaquette with no complex numbers

T-P4-5 asserted that CP violation is a chirality of loop angles; here is the computation, executed entirely in real rotation blocks. Build the CKM matrix in the standard parameterization with angles $(s_{12}, s_{13}, s_{23}) = (0.2250,\ 0.00369,\ 0.04182)$ and the parent program's first-order phase $\delta = 2\pi/5$, but replace every complex entry z by its block $B(z)$ of (4) — so $e^{-i\delta}$ is the real rotation $R(-\delta)$, and the matrix is a real 6×6 array of blocks. The rephasing-invariant quartet is the plaquette product

$$P \;=\; B(V_{us})\,B(V_{cb})\,B(V_{ub})^{\mathsf{T}}B(V_{cs})^{\mathsf{T}}, \qquad J_{\text{CKM}} \;=\; P_{21}, \tag{33}$$

conjugation being block *transpose* (orientation reversal, Proposition 10.3) and the imaginary part being the antisymmetric (sine) component. The numbers: $\delta = 2\pi/5$ gives $J_{\text{CKM}} = 3.21\times 10^{-5}$, against the measured $(3.12^{+0.13}_{-0.12})\times 10^{-5}$; the achiral values $\delta \in \{0, \pi\}$ give $J_{\text{CKM}} = 0$ identically; and re-zeroing all six family dials by random angles ($\varphi_{ij} \mapsto \varphi_{ij} + \alpha_i - \beta_j$ in block form) leaves P_{21} unchanged to machine precision — rephasing invariance *as* holonomy invariance, watched happening. At no step does the symbol i occur: CP violation is an area-like real invariant of a real texture, exactly as T-P4-5 states.

35.6 Worked example VI: the dial-free door, in matrices

Lemma 12.1 in coordinates, for the two cases that matter. *Weight one:* under $R(\alpha)$, the Majorana doublet transforms as

$$\begin{pmatrix} a^2-b^2 \\ 2ab \end{pmatrix} \;\longmapsto\; \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} a^2-b^2 \\ 2ab \end{pmatrix}, \tag{34}$$

so the only invariant self-pairing is $a^2+b^2 = \psi\bar{\psi}$: a charged fermion can have a Dirac mass and nothing else, and lepton number — winding number — is conserved at every vertex. *Weight zero:* the rotation is the identity, all three pairings are invariant, and the reality condition *is* the Majorana condition: the $\Delta L = 2$ self-pairing is legal exactly here. The portal arithmetic closes the story: the unique fiber-neutral trilinear touching the door is LHN with weights $(-\frac{1}{2}) + (+\frac{1}{2}) + 0 = 0$. In the parent program this one-line weight count is the constitutional basis of the seesaw and of TeV-scale leptogenesis (Arisaka, 2026, A10-Cosmos); here it serves as the sharpest possible illustration that the circle's conservation laws are *winding arithmetic* — integers, not analysis.

Close of Part V, in plain terms. Positive frequency is holomorphy: at one clock it is the Hardy property of the analytic signal; demanded of all inertial clocks it is analyticity in the forward tube; and granted light's full conformal symmetry it becomes the division of twistor space into its two halves — the same splitting, generated by the same derived J, drawn at three magnifications of symmetry. The worked examples made each level concrete enough to redo by hand. What has been proved so far, however, lives in the free and quasi-free world; the next Part patrols the boundaries — interactions, the infrared, horizons — and states exactly where the derivation extends, where it is conditional, and — at the infrared, with the door theorem — where a boundary long recorded as open turns out to close.

Part VI
Extensions and Boundaries

Interacting matter; the infrared door opened; three superselections, one custody principle

36 Part VI in plain English: past the free field, and where the account stops

The theorems of Part V are kinematic and quasi-free. This Part carries them as far as they will honestly go and then marks the place where they stop.

Interacting matter is reached by the asymptotic route, and the completion is conditional: it rests on a mass gap and on asymptotic completeness, which are the standard scattering hypotheses and are named as hypotheses rather than absorbed into the proof. The infrared case is the harder one, because massless interacting theories have no gap. The theorem proved here closes that door at the rigorous-model level, exactly at the van Hove stratum, and the full relativistic dressing theory is named as the rung above and left open.

The Part ends by noticing that superselection has now been met three times — at the charge fiber, in the infrared sectors, and in records — and that the three are one structure met on three strata of physics. Each is written up in the same four items, and the pattern they share is then proved from the theorems already in hand rather than asserted, which is the difference between a pattern and a coincidence.

37 Interacting matter: the asymptotic completion

Theorem 37.1 (**T-P4-8**: the interacting completion; conditional)**.** *Let interacting bulk matter satisfy: (H1) a mass gap; (H2) asymptotic completeness; (H3) the regularity of a Wightman or Haag–Kastler framework. Then the physical Hilbert space is canonically the Fock space over the asymptotic real solution spaces equipped with the derived J of T-P4-2/T-P4-3: the interacting theory's complex structure* is *the free structure of its asymptotic frame, and interactions contribute vertices — real texture data — and no complex structure of their own.*

Proof (by assembly). Under (H1) and (H3), Haag–Ruelle theory (Haag, 1992) constructs, from almost-local smearings of the interacting fields with one-particle-supported test functions, creation operators whose strong limits as $t \to \pm\infty$ exist (cluster estimates supply the Cauchy property) and satisfy free-field commutation relations at the physical mass; the generated in/out spaces are Fock over free one-particle spaces, which are exactly the real-data objects of Part IV with the polar-phase J. Under (H2) the in/out spaces exhaust the physical space and the Møller operators transport J

onto it. Vertices enter only through the S-matrix and the texture (moduli and fiber angles), both real. ■

Physics intuition. An interacting world is still interrogated by free probes: what a collider registers, long before and long after the collision, are freely flying quanta, and it is *their* dials and clocks that define the bookkeeping. The theorem says the bookkeeping suffices: the complex structure never belonged to the interaction — it belonged to the asymptotic quanta, where the free derivation already owns it. For confined sectors the same statement attaches to hadrons: the particles instruments actually meet, which are gapped.

The hypotheses, weighed. (H1), the mass gap, is doing two jobs: isolating the one-particle shell so that quasi-local creation operators can be tuned to it, and guaranteeing the cluster estimates that make the $t \to \pm\infty$ limits Cauchy. (H2), asymptotic completeness, is the statement that scattering exhausts the theory — unproven in general four-dimensional field theory for anyone, and the honest hinge of the theorem: without it the derived J owns the scattering sector and is silent on a hypothetical non-scattering remainder. (H3) is bookkeeping regularity. What the theorem buys under these hypotheses is worth restating: not an approximation scheme but an *identification* — the interacting Hilbert space, whatever its internal complexity, carries exactly the asymptotic frame's complex structure, so interactions renormalize masses, couplings, and states, but never the meaning of i.

Physics intuition. A storm at sea does not change what "clockwise" means. The interaction region may be arbitrarily violent; the complex structure is an orientation convention established where the water is calm — at $t \to \pm\infty$ — and transported through the storm by unitarity. That the two calm regions agree (in against out) is asymptotic completeness's real content here, and it is also why the confined sector poses no special problem: one reads the orientation off the hadrons, which are the calm water.

38 The infrared door: sectors are ports

Where masslessness and interaction meet, the asymptotic particle notion itself becomes subtle. The charged particle cannot shed its soft-photon cloud; its sharp mass shell dissolves (the *infraparticle*); and clouds belonging to different velocities are superselected: for cloud profiles $c_v/\sqrt{\omega}$ the overlap of two dressings carries the exponent

$$\left|\langle\, \mathrm{cloud}_v \,|\, \mathrm{cloud}_{v'} \rangle\right| \;=\; \exp\!\Big(-\tfrac{1}{2}\int_{\epsilon}^{\Lambda} \frac{|c_v - c_{v'}|^2}{\omega}\, d\omega\Big) \;=\; \Big(\frac{\epsilon}{\Lambda}\Big)^{|c_v - c_{v'}|^2/2} \;\xrightarrow[\epsilon\to 0]{}\; 0, \qquad (35)$$

logarithmically divergent in the exponent and power-law vanishing in the overlap — the Bloch–Nordsieck signature (Bloch and Nordsieck, 1937). Rigorous collision theory exists for the massless quanta themselves (Buchholz, 1977), and dressed-state constructions (Chung, 1965; Faddeev and Kulish, 1970) organize the charged sectors, rigorously so in model systems. Two structural facts govern the reading here.

Lemma 38.1 (The cloud never touches J)**.** *A coherent dressing is an affine symplectic map $f \mapsto f + d$: it shifts first moments, leaves every covariance invariant, and hence leaves the complex structure $J = 2\Gamma\sigma$ of the sector's Gaussian states exactly unchanged.*

Proof. Covariances are second central moments; an affine shift changes only the center. $J = 2\Gamma\sigma$ is built from the covariance and the (fixed) symplectic form. ■

The infrared therefore threatens the *particle* notion, never the derivation of i; and the sector label — which infrared background an instrument treats as vacuum — is operational boundary data, of exactly the kind AX-3 assigns to instruments. The remainder of this section makes that reading precise and proves it, at exactly the level the rigorous models support. The result is the *door theorem*, T-P4-9: the corner where this account was long weakest — and where the twistor school's commitments are strongest, massless fields being its native habitat — closes, and closes in a way that strengthens the thesis, because what the infrared turns out to be made of is real, classical, coherent bookkeeping.

Physics intuition. An electron drags an unpayable debt of arbitrarily soft light, and two electrons moving differently drag inequivalent debts — so "the same state, boosted" splits into sectors no local operation connects. But notice what the debt is made of: a classical, coherent, real displacement of the field. Shifting a pendulum's rest point does not change what "a quarter period ahead" means. The infrared rearranges the furniture; it does not touch the clock.

The model ladder. The proof of the door theorem lives on a ladder of three rungs, and the theorem's regime is exactly the ladder's. The bottom rung is the *van Hove stratum* — a massless field linearly coupled to a fixed (recoil-free) source, $H = \mathrm{d}\Gamma(\omega) + \phi(g)$ — where everything is exactly solvable and the lemmas below are the whole content. The middle rung is the *Nelson and Pauli–Fierz models* — a nonrelativistic particle coupled to the massless scalar field or to the quantized radiation field — where the infraparticle structure is theorem-grade: the fixed-momentum Hamiltonian has no ground state in Fock space (Fröhlich, 1973); the dressed one-particle states exist, with Hölder regularity in the total momentum (Pizzo, 2003); the mass shell survives removal of the ultraviolet cutoff (Bachmann et al., 2012); and the infraparticle scattering states of nonrelativistic QED carry exactly the velocity-superselected clouds the door theorem classifies (Chen et al., 2010). The top rung is *full relativistic QED*, where the dressing theory (Chung, 1965; Faddeev and Kulish, 1970) is semi-rigorous and the sector observables are the asymptotic fluxes of Buchholz (1986); the theorem's reading carries verbatim there, and its status is displayed, not claimed — the field's own frontier, inherited with its name on it.

Definition 38.2 (Coherent sectors)**.** Let $W(h)$, for h in the real test space, generate the Weyl algebra of the asymptotic massless field. A *coherent dressing* by a profile f is the algebra automorphism $W(h) \mapsto e^{i\,\mathrm{Im}\langle f,h\rangle}\, W(h)$. Its GNS representation over the vacuum state is Fock if and only if f is square-integrable, and two coherent representations with profiles f, f' are unitarily equivalent if and only if $f - f' \in L^2$. A *sector* is an equivalence class of profiles; its label is c-number data (the flux configuration at infinity), and the label is invariant under every quasi-local operation.

Lemma 38.3 (The cloud leaves Fock space; velocities are superselected)**.** *In the massless models of the ladder, the dressing cloud of a charge moving with velocity v ($|v| < 1$) has the low-frequency profile*

$$f_v(k) \;=\; \frac{-\,g}{\sqrt{2|k|}\;\;|k|\,(1 - v\cos\theta)}, \qquad |k| \le \Lambda, \tag{36}$$

with θ the angle between k and the velocity. Then, with infrared cutoff ϵ: (i) $\|f_v\|^2 \;=\; \dfrac{2\pi g^2}{1-v^2}\,\ln\dfrac{\Lambda}{\epsilon} + O(1) \xrightarrow[\epsilon\to 0]{} \infty$, so in the cutoff-removed limit $W(f_v)$ is not implementable on Fock space and the dressed state generates a new coherent sector; (ii) for $v \ne v'$, $\|f_v - f_{v'}\|^2 = c(v,v')\,\ln(\Lambda/\epsilon) + O(1)$ with $c(v,v') > 0$, so the sector overlap obeys the power law (35) and distinct velocities are exactly orthogonal in the limit: velocity superselection, by the same mathematics that superselected charge — an unremovable divergence in a relative-phase integral.

Proof. For (i): $|f_v(k)|^2 = g^2/(2|k|^3\,(1 - v\cos\theta)^2)$, so in spherical coordinates

$$\|f_v\|^2 \;=\; \frac{g^2}{2}\int_\epsilon^\Lambda \frac{dk}{k}\int d\Omega\,\frac{1}{(1-v\cos\theta)^2} \;=\; \frac{g^2}{2}\,\ln\frac{\Lambda}{\epsilon}\;\cdot\;2\pi\int_{-1}^{1}\frac{du}{(1-vu)^2} \;=\; \frac{2\pi g^2}{1-v^2}\,\ln\frac{\Lambda}{\epsilon},$$

using $\int_{-1}^{1}(1-vu)^{-2}\,du = 2/(1-v^2)$. For (ii): the difference profile carries the same $|k|^{-3/2}$ infrared singularity with the angular factor $(1-v\cos\theta)^{-1}-(1-v'\cos\theta)^{-1}$, nonzero on a set of positive measure whenever $v \ne v'$; the same radial integral gives $c(v,v') = \pi g^2\int_{-1}^{1}\big[(1-vu)^{-1} - (1-v'u)^{-1}\big]^2 du \,>\, 0$. Substituting into the coherent overlap formula reproduces (35) with $|c_v - c_{v'}|^2$ read off as $c(v,v')$: the overlap is $(\epsilon/\Lambda)^{c/2}$, vanishing as the cutoff is removed. Disjointness is then the L^2 criterion of Definition 38.2. ∎

Lemma 38.4 (The dressed generator is free-plus-c-number)**.** *On the van Hove stratum, mode-wise: the coupled generator $H = \omega\,a^*a + g\,(a + a^*)$ is diagonalized by the displacement $W(d)$ with $d = -g/\omega$, and*

$$W(d)^\dagger\,H\,W(d) \;=\; \omega\,a^*a \;-\; \frac{g^2}{\omega}; \tag{37}$$

in the multimode case, $W(d)^\dagger H\,W(d) = \mathrm{d}\Gamma(\omega) - \langle g,\,\omega^{-1}g\rangle$. The quadratic part — the only datum the polar-phase construction of T-P4-2 reads — is exactly the free generator, in every sector.

Proof. $W(d)^\dagger\,a\,W(d) = a + d$, so

$$W^\dagger HW \;=\; \omega\,(a^* + d)(a + d) + g\,(a + a^* + 2d) \;=\; \omega\,a^*a \;+\; (\omega d + g)(a + a^*) \;+\; \omega d^2 + 2gd,$$

and the choice $d = -g/\omega$ cancels the linear term, leaving the c-number $-g^2/\omega$. Together with Lemma 38.1 — the affine shift leaves every covariance, hence $J = 2\Gamma\sigma$, exactly invariant — the sector's derived complex structure is the polar phase of the *same* quadratic generator in every sector. ∎

Theorem 38.5 (**T-P4-9**: the infrared door — sectors are ports; J is sector-blind)**.** *In an interacting theory with massless quanta, on the model ladder at the levels stated there: (i) the asymptotic soft-cloud configuration — the flux/velocity sector — labels mutually disjoint representations of the* *asymptotic field algebra, and the label is a c-number port assignment at AX-3: which infrared background an instrument treats as "vacuum" is operational boundary data, with the same standing as the horizon boundary condition of the black-hole case; (ii) within each sector the dressed asymptotic dynamics is free-plus-c-number, and its polar phase supplies the* same *derived J in every sector — sector-blind, because its two sources, the circle and the clock, are global; (iii) hence the derivation of this paper extends to the infrared: the sector multiplicity is dictionary data, the complex structure passes through untouched, and the two never mix. The infrared problem is a problem about which vacuum dressing an observer stands in — not a problem about where i comes from.*

Proof (by assembly). At the van Hove stratum the theorem *is* the three lemmas: clause (i) is Lemma 38.3's disjointness together with the invariance of the label under quasi-local operations (Definition 38.2) — a multiplicity of admissible vacua, labeled by c-number boundary data, unresolvable from within, which is precisely the kind of datum AX-3's dictionary owns, and precisely how the horizon vacua are placed in §39 below; clause (ii) is Lemma 38.4 together with Lemma 38.1; clause (iii) follows from (i) and (ii). At the Nelson/Pauli–Fierz rung, the imported theorems supply exactly the two existence facts the lemmas need — the dressed one-particle states exist (Bachmann et al., 2012; Pizzo, 2003), and their clouds carry the profile (36) with velocity superselection (Chen et al., 2010; Fröhlich, 1973) — and the lemmas, being statements about coherent dressings as such, apply verbatim to the states those theorems construct. At full QED the dressing theory is semi-rigorous (Chung, 1965; Faddeev and Kulish, 1970), the sector label is physical — by Gauss' law it is an asymptotic flux observable (Buchholz, 1986) — and the reading carries with its status displayed. ■

Verification. Four numerical checks accompany the door theorem in the verification suite (§46). The cloud norm's logarithmic slope matches the analytic coefficient $2\pi g^2/(1-v^2)$ of Lemma 38.3 to 8×10^{-6} relative. The measured sector-overlap power matches the quadrature coefficient $c(v,v')/2$ to 10^{-5} relative, the overlap falling from 10^{-14} to 10^{-53} over six decades of cutoff at strong coupling and relativistic velocity separation. The exact dressed ground state of a coupled two-mode van Hove model carries $\max|J - J_{\text{free}}| = 5 \times 10^{-15}$ while its infrared-tower sector label $\|d\|^2$ grows without bound at the analytic slope. And the conjugation (37) holds on the truncated Fock space to machine precision, the residual falling as pure truncation error as the truncation is raised.

The frontier, named. What T-P4-9 does not close is the top rung: full relativistic QED, where the existence theory of the dressing is the field's own open problem (the Faddeev–Kulish program). This account inherits that problem by name rather than hiding it — and notes what kind of problem it is. It is an *existence* question about dressed states, not a question about the origin of i: clauses (ii) and (iii) are statements about coherent dressings as such and wait, verbatim, on any dressing the full theory constructs. The waiting can itself be stated as a conditional theorem. Suppose full QED admits a dressing theory satisfying: *(D1) coherent form* — the physical charged states are Weyl displacements of Fock states by c-number soft-photon profiles; *(D2) flux sector labels* — two dressings are L^2-equivalent if and only if they carry the same Gauss-law flux class; *(D3) free-plus-c-number dressed asymptotic dynamics* in each sector. Then clauses (i)–(iii) of T-P4-9 hold verbatim in full QED, by the same three lemmas. (D1)–(D3) are exactly what the semi-rigorous constructions exhibit and what the rigorous program is built to deliver; the residue is purely their existence.

Physics intuition. Think of the infrared as a port fee. Every instrument that does business at spatial infinity must declare a convention: which configuration of arbitrarily soft light it will call "empty." Two instruments moving differently make different declarations, and no local transaction can convert one declaration into the other — that is all the superselection is. But the fee is paid in a real currency: a classical, coherent displacement of the field. Nothing about the declaration touches the clock that defines "a quarter period ahead." The tellers differ; the currency is the same.

For the reader of Part V. The door theorem completes a movement this paper began at the Hardy level. The massless conformal sector is the natural home of the bridge (T-P4-6, T-P4-7), and its interacting extension was the one corner where this account thinned — the corner, moreover, where a twistor-school reader keeps the deepest commitments, massless fields being the program's native habitat. T-P4-9 says the thinning was bookkeeping: what dissolves in the infrared is the Fock mass shell — a statement about which states are reachable from which — while the splitting the bridge is made of, positive against negative frequency, passes through every sector untouched. The reader who came for the twistor geometry can take the door theorem as the statement that nothing in the infrared bends the geometry of $\mathbb{PT}^{\pm}$: the division of twistor space is sector-blind too, because it is generated by the same J.

39 Three superselections, one custody principle

The derivation has now met superselection three times, and this section writes the three encounters up as what they have turned out to be: one structure, met on three strata of physics, each time by proof. For uniformity, each instance is recorded in the same four items, in the same order: the *arena*; the *unreadable datum*, with the mechanism that protects it; the *sector label*, and where it lives; and the *status of J*. The pattern the three instances share is then stated once, as a proposition, and proved from the theorems already in hand.

First instance: charge — the port at the fiber. *Arena:* the compact phase circle of T-P4-1, carried by every charged field. *Unreadable datum:* the absolute fiber angle. A four-dimensional instrument couples only to angle *differences* and to windings; the absolute angle is unobservable by construction, and the protection mechanism is geometric — the same compactness that quantizes charge hides the origin of its dial. Relative phases between sectors of different winding are therefore unmeasurable: this is charge superselection, in the form first isolated by Wick et al. (1952), here with a geometric mechanism supplied. *Sector label:* the winding number $n \in \mathbb{Z}$ — the ledger's $M = 1$ integrality — a global topological datum that no quasi-local operation changes; which winding sector an instrument's bookkeeping addresses is dictionary data at AX-3. *Status of J:* within each sector the circle supplies J_Q, and on shell the clock's polar phase agrees with it exactly (T-P4-3), so one derived J acts identically on every sector (T-P4-4). The sectors differ in an angle no one can read; the quarter turn is the same everywhere.

Second instance: horizon vacua — the port at the horizon. *Arena:* a stationary black-hole exterior. *Unreadable datum:* the interior partner correlations — the modes behind the horizon that purify the exterior state. The protection mechanism is causal: the horizon is a one-way boundary, the exterior state is the partial trace across it, and thermality at the standard temperature is produced by the trace itself — the GG-6 boundary projection (Arisaka, 2026, A5-GDOS) evaluated at a horizon, in the horizon-as-port reading of Arisaka, 2026, C3-BH. *Sector label:* the boundary condition at the horizon — Boulware, Unruh, Hartle–Hawking (Hawking, 1975; Unruh, 1976; Wald, 1994) — a declaration of which port state the instrument's bookkeeping assumes: dictionary data at AX-3, not bulk ambiguity. *Status of J:* on the stationary exterior the Killing-frame J is unique by the positivity selection (Kay, 1978; Kay and Wald, 1991); port choices repopulate the states and never touch the complex structure. The rotating case — where the ergoregion defeats the exterior selection and the anchor moves to the asymptotic frame, superradiance becoming a symplectic pumping channel — and the collapse epoch — which factorizes into a universal thermal channel times a decaying transient — extend this instance at the model level in the parent program (Arisaka, 2026, A8-QM, entries R14–R15; the full-length treatment is the black-hole companion, Arisaka, 2026, C3-BH).

Third instance: infrared clouds — the port at infinity. *Arena:* interacting massless matter, on the model ladder of §38. *Unreadable datum:* the soft-cloud profile at infinity, relative to another sector's. The protection mechanism is analytic: the relative-phase integral between clouds diverges logarithmically (Lemma 38.3), so the inter-sector overlap vanishes as a power of the cutoff — an unremovable divergence standing exactly where, for charge, the compactness stood. *Sector label:* the velocity/flux class — by Gauss' law an asymptotic flux observable (Buchholz, 1986); which infrared background counts as "vacuum" is an instrument's port declaration at AX-3 (T-P4-9(i)). *Status of J:* sector-blind, by Lemma 38.1 and Lemma 38.4 (T-P4-9(ii)): the dressing is affine, the quadratic generator is untouched, and every sector's polar phase is the same J.

Proposition 39.1 (The custody principle)**.** *In each of the three instances: (a) the sector label is a c-number datum conjugate to a* global *structure — the total winding, the horizon interior, the field at infinity — and is invariant under every quasi-local operation; (b) the relative phase between sectors is unobservable, the obstruction being in each case an unremovable contribution to a relative-phase integral (the compact holonomy; the traced interior; the logarithmically divergent cloud); (c) the derived J commutes with the sector decomposition and is identical in every sector. Consequently the superselection structure is owned entirely by the AX-3 dictionary — all three labels are port assignments — and the complex structure entirely by the geometry:* the dictionary owns the sectors; the geometry owns the i.

Proof. Instance by instance. *Charge:* (a) is the topological invariance of the winding (T-P4-1); (b) is the unreadability of the absolute angle — the geometric form of the superselection rule of Wick et al. (1952); (c) is T-P4-3 with T-P4-4: one J on shell, acting identically on every winding sector. *Horizon:* (a) the port condition is set at the horizon, a global causal boundary, and no exterior quasi-local operation reaches through it; (b) the partner phase sits behind the partial trace; (c) the Killing-frame J is unique on the stationary exterior (Kay, 1978; Kay and Wald, 1991), independently of the port choice. *Infrared:* (a)–(c) are T-P4-9, through Lemma 38.3, Lemma 38.1, and Lemma 38.4. ■

That division of custody is, in one sentence, this preprint's account of the quantum formalism — and it is worth pausing on why the pattern recurs. In all three instances the superselected quantity is conjugate to something global, and in all three the local physics is untouched: superselection is the boundary's bookkeeping of what no local operation can reach. The i, by contrast, is built from structures — the circle's plane of rotation, the clock's orientation — that are shared *within* every sector and identical *across* them. A theory whose complex structure came from anywhere else would have no reason for this immunity; on the present account the immunity is clause (c) of a proposition, proved three times, on three strata, by three different mechanisms guarding the same kind of secret. One scope remark belongs here. The proposition is deliberately confined to the superselection triple — the three instances in which an unreadable *relative phase between sectors* exists to be owned. The parent program's rotating anchor and collapse channel (entries R14–R15 of Arisaka, 2026, A8-QM) extend the same division of custody to five instances at the model level, but they are not superselection structures: what the dictionary owns there is a frame anchor and an in-channel, not a sector label, and the proposition's clauses apply to them in the looser currency of custody rather than the strict currency of sectors. The slogan survives unchanged in both currencies: the dictionary owns what varies; the geometry owns the i.

Close of Part VI, in plain terms. The boundaries are now mapped — and one of them turned out to be a door. Interacting gapped matter inherits the asymptotic frame's i, conditionally on the field's own standard hypotheses. The infrared, long this account's named open corner, is crossed at the rigorous-model level: what dissolves there is the particle's sharp mass shell — a fact about sector labels — while the complex structure passes through every sector untouched (T-P4-9). And the three superselections the account has met — charge, horizon vacua, infrared clouds — are proved to be one pattern (Proposition 39.1): an unreadable relative phase, a port assignment at the dictionary, an untouched J. The dictionary owns the sectors; the geometry owns the i — no longer a slogan but a proposition with three instances. With the theorems proved and the boundaries drawn, the next Part steps back to judge: what the result signifies, to whom, what is genuinely new in it, how the three great roads compare, and what would prove it wrong.

Table 3: The three superselection structures this paper meets, written uniformly: an unreadable relative phase between sectors, an operational sector label (a port assignment at AX-3), and a derived J that passes through untouched — the pattern stated and proved as Proposition 39.1.

Instance	**Unreadable datum**	**Sector label (port)**	J **status**
Charge	the absolute fiber angle	winding number n	J_Q per sector; one J on shell (T-P4-3)
Horizon vacua	the interior partner modes	boundary condition at the horizon	Killing-frame J unique on the exterior
Infrared clouds	the soft profile at infinity	velocity/flux sector	T-P4-9: ports; J sector-blind (theorem, model level)

Part VII
Discussions

Significance and uniqueness of the result; the three roads compared, string theory included; predictions for falsification; the open questions; and the summary of the case

40 Part VII in plain English: what the result is worth, and what would end it

A paper that has proved its theorems still owes an account of what they are worth. This Part gives that account at length rather than in a closing paragraph, and most of it consists of comparisons the paper could have skipped.

It states what is claimed and in what regime. It sets the result beside string theory and the twistor program on the axes where the three intersect, following each program's own literature and adjudicating nothing. It then makes the twistor comparison aspect by aspect, marking the seam where the two accounts meet and naming the columns where the twistor program is ahead — the nonlinear constructions, and the amplitudes.

Then the falsifiers and the open questions. There are six of the latter, in decreasing order of consequence, and the first two are load-bearing. The falsifiers are stated as measurements rather than as attitudes. None of the open questions threatens the theorems as stated; each bounds the regime the theorems own. A reader deciding how much weight this argument can carry should read that section before reading the conclusions.

41 Significance of the result

Three communities keep a stake in the question this paper answers, and the significance reads differently to each; a fourth and a fifth stake, less obvious, are argued below.

To quantum foundations, the result relocates a century-old postulate. Since von Neumann's axiomatization, the complexity of the state space has been the first assumption of the formalism — defended, when defended at all, by its consequences. The reconstruction program sharpened the defense into theorems: complex quantum mechanics is selected among operational theories by information axioms. What no result of that kind supplies is a *referent* — a thing in the world that the complex structure is the structure *of*. This paper supplies the referent (a circle and a clock, both derived), a uniqueness clause (positivity on the static arena), and an experimental fingerprint read in reverse: the network-Bell exclusion of real quantum mechanics, which the foundations literature received as a surprising theorem about formalisms, becomes the expected signature of two globally shared real sources — the one experimental fact this account could not survive without, found already in hand. A postulate with a referent, a uniqueness proof, and a fingerprint is no longer a postulate in the working sense; that conversion is the significance, whatever one thinks of the parent program.

To the twistor school, the result is a duality with a gift at each end. Read toward twistor space, it supplies the motivation the program has done without for fifty years: holomorphy is what causal positivity looks like at conformal symmetry, so the school's central structure acquires a physical source, and its founding question — why should nature be holomorphic? — acquires an answer built from the school's own theorems. Read back from twistor space, it explains the program's phenomenology of strengths: why massless fields are the native habitat (only conformal fields complexify every timelike direction at once); why the massive sector resists (it stops at the tube level, where the clock works but the conformal completion is unavailable); and why gauge charge always needed separate apparatus (the circle's half of the derivation is invisible to ray-space geometry, by the division of labor of T-P4-3). A program that learns, from outside, the exact boundary of its own naturalness has gained something no internal development could give it.

To the parent program, the result closes a ledger. The GG-Theory cohort derives its physics from real axioms, and until this line of work its own formal apparatus carried one unexamined complex ingredient — the state spaces of its boundary theory. The real-bulk series settled the accounts: no complex primitive anywhere in the constitution, with the residues named as analytic frontiers rather than conceptual debts. And the closure is not bookkeeping alone: three results of the cohort that looked separate — the antiparticle pairing, the lattice-locked CP phases, the forced Majorana door — are exhibited here as one family, consequences of how the circle and the clock share, twist, and withhold the single derived J. A framework whose scattered rigidities turn out to be one mechanism has learned something about itself.

To the history and philosophy of physics, the significance is a case study of unusual purity. Wigner's "unreasonable effectiveness" has always wanted load-bearing instances examined one at a time; the complex numbers are arguably its sharpest instance, and on this account the instance converts from unreasonable to unrecognized: the mathematics was effective because it was, unknowingly, a portrait of instantiated structure. Each brute fact traded for a consistency condition — quantization for compactness, the antiparticle for a handshake, antiunitarity for orientation, holomorphy for positivity — is a data point in the oldest methodological question physics has, and six data points from one mechanism is a pattern.

Finally, the significance of exposure. Foundational accounts are cheap when nothing can refute them; this one lists its failure modes and would not survive them. A demonstration that independent sources can carry unshared complex structures; a measured drift of the flavor phases off their lattice completion; an established Dirac nature for the light neutrinos; a derivation of the circle and the clock *from* holomorphy, reversing the arrow — any of these retires or inverts the account. In foundations, where claims commonly retreat to interpretation when pressed, an answer that stakes named ways to be wrong is significant partly *because* it can lose.

42 Uniqueness and innovation

To our knowledge, no other framework derives both complex structures of physics from stated axioms, and the survey substantiating that sentence is worth a paragraph. The textbook tradition postulates the complex Hilbert space outright and correctly declines to argue. The reconstruction theorems select complex quantum mechanics among operational rivals — an existence-of-necessity result, not a source. The real-formulation program, at its most serious in Stueckelberg, showed a real presentation survives for single systems and stalls at composites — exactly where the shared-J quotient of T-P4-4 now locates the obstruction, and where experiment later dug the grave. Geometric quantization *chooses* Kähler polarizations with celebrated non-uniqueness; the present account is, in that language, a physics that forces the polarization. The twistor reading takes holomorphy as constitutive; the string program takes the worldsheet's complex structure as given with the formalism. Each of these is a coherent position; none of them answers, or attempts, the question of what the imaginary unit is made of.

The specific innovations claimed as new are five, and each can be stated with its precise delta from the prior art. *First*, the two-sources decomposition with its coincidence theorem: that the gauge phase and the state-space i have different real sources, agreeing on charged sectors exactly under the Feynman–Stückelberg pairing — so that antiparticles, imposed in every textbook as a positivity condition on the output of quantization, are derived one level down as the unique meshing of two independently forced structures. *Second*, the discrete-symmetry table as orientation bookkeeping: unitarity versus antiunitarity, assigned by Wigner's theorem case by case, is here read off from which of the three real orientations an operation reverses — antiunitary if and only if the clock is touched. *Third*, the identification of the constitutionally selected J with the generator of the Hardy splitting, which converts the analytic signal from an engineering device into the boundary form of the state space's own complex structure. *Fourth*, the direction-reversal at twistor space: the division $\mathbb{PT}^+/\mathbb{PT}^-$, present in the school's literature from the beginning as a property of primitively complex geometry, re-derived as the conformal completion of causal positivity — the same seam, sewn from the other side. *Fifth*, the arithmetic layer: with the integer ledger's single circle and five-block, the CP-violating loop holonomies are lattice-valued, making CP violation a *discrete* chirality — a statement unavailable to any framework in which phases have no geometry to be locked to.

Three further structural readings are claimed as new framings with consequences — one of them now proved in full: the uniform mechanism of the three superselections, now stated and proved as the custody principle (Proposition 39.1), with its infrared instance at theorem level (T-P4-9); the port classification of the vacuum and infrared multiplicities as operational dictionary data rather than bulk ambiguity — for the infrared no longer a reading but a theorem; and the compression account of the historical sequence, which turns the four-century gap between Cardano and Schrödinger from an irony into an explanation.

Equally deliberate is what is *not* claimed. The Paley–Wiener theory, the stationary uniqueness theorems, Haag–Ruelle scattering, the Penrose transform and its cohomology, the superselection formalism — every load-bearing import is credited at its point of use and shelved in the reading guide, and the paper's contribution is stated as assembly plus direction: imported bricks, a new building, and an arrow of explanation that the bricks' own literature does not contain. The nearest historical anticipations deserve the same candor, because each held a piece: Hamilton had the real presentation and lacked the selection; Wu and Yang had the holonomy dictionary and lacked the dimension; Stueckelberg had the real Hilbert space and lacked the sharing; Penrose had the division and read the arrow the other way. Novelty, in a subject this old, is precisely a statement about which gap remained — and the gap this paper fills is the conjunction: selection, dimension, sharing, and arrow, at once.

Uniqueness, last, in the structural sense: within the parent framework this account could not have been otherwise. The arena is forced (Theorem 0), the circle's quarter-turn is unique up to the orientation the clock fixes (T-P4-1, T-P4-3), the clock's polar phase is unique by positivity (T-P4-2), and the sharing is automatic because the sources are global (T-P4-4). Nothing was available to tune. A derivation with no adjustable element either connects to the world or breaks; §48 lists where to strike it.

43 Three roads: string theory, twistor theory, and GG-Theory

It is orienting to place a third program beside the two this paper has compared, and the house rule of the cohort applies: structural contrasts only, stated so that each program's own logic is respected. String theory reaches its extra dimensions by an internally compelled path — worldsheet consistency and anomaly cancellation fix the dimension and the gauge structure almost rigidly (Green and Schwarz, 1984) — and its relation to complex structures is intimate but different in kind from either program here: the worldsheet is a Riemann surface, so two-dimensional holomorphy is a primitive of the formalism, while the target-space quantum i is inherited from ordinary quantization. The three programs thus place the complex numbers in three different positions: primitive on the worldsheet (strings), primitive in the geometry of ray space (twistors), derived from a circle and a clock (here). Table 4 records the comparison on the axes where the three genuinely intersect; the twelve-aspect twistor–GG-6 table of §45 remains the detailed instrument, and the string column is offered for orientation, not adjudication — the two economies place their irreducible premises at opposite ends of the energy axis, and the reader is left to weigh them.

44 What is claimed, and in what regime

Claimed, as theorems at the kinematic quasi-free level (T-P4-1 through T-P4-7): the gauge phase is circle geometry; the state-space i is the polar phase of time, unique by positivity on the static arena; the two coincide on charged sectors exactly under the particle–antiparticle pairing; the discrete-symmetry table is orientation bookkeeping; CP phases are lattice-locked loop holonomies; the multipartite complex structure costs nothing beyond the shared J; and positive frequency is holomorphy at the Hardy, tube, and twistor levels. Claimed conditionally, on the standard scattering hypotheses (T-P4-8): the interacting completion. Claimed at the rigorous-model level (T-P4-9): the infrared door — sectors are ports, the derived J sector-blind — exact at the van Hove stratum

Table 4: The three roads, on the axes where they intersect. Entries follow each program's own literature; no adjudication is intended. Seven aspects, one per row: where the complex numbers enter, how many extra dimensions are carried and of what kind, why four dimensions, what is assumed about quantum mechanics, how mass arises, where each program's premises sit, and whether the complex-number question is posed at all. The three are set side by side because they are usually compared on their outputs, and what separates them is visible earlier than that, in what each takes as primitive.

Aspect	**String theory**	**Twistor theory**	**GG-Theory**
Where $\mathbb{C}$ enters	worldsheet complex structure (primitive); target-space i from quantization	holomorphy of $\mathbb{PT}$ (primitive)	derived: circle + clock (T-P4-1–3)
Extra dimensions	six/seven, spatial, compactified; moduli and landscape downstream	none (an auxiliary complex ray space)	two, non-spatial, forced; no moduli
Why four dimensions	selected by compactification	built into the formalism	derived (4+1+1, 4D boundary)
Quantum mechanics itself	presupposed	presupposed	boundary law derived from the bulk (Arisaka, 2026, A5-GDOS)
Mass	natural (compactification scales)	a named hurdle	scale fiber + gap (T-P4-2, T-P4-8)
Premises located at	trans-Planckian consistency	conformal geometry of light	accessible-energy axioms
Complex-number question	not addressed	posed at full strength	answered in calibrated halves

and theorem-grade at Nelson/Pauli–Fierz on imported hypotheses. Open and named: the full-QED completion of the door (§38), now a conditional theorem on displayed dressing hypotheses; the exact-Kerr Hadamard analysis and the full gravitating collapse case, whose model-level cores are closed in the parent treatment (Arisaka, 2026, A8-QM, entries R14–R15); and the nonlinear twistor constructions beyond the free correspondence. The register throughout is the cohort's calibrated one: derived where provable, conditional where conditional, open where open.

45 Twistor theory and GG-Theory: a structural comparison

The two programs are natural companions rather than rivals: twistor theory supplies the deepest available mathematics of the massless conformal sector, and GG-Theory supplies real sources for the structures twistor theory takes as given. Table 5 records the comparison aspect by aspect, in the calibrated register: where the twistor program itself names an open hurdle, the table says whether the present framework addresses it, inherits it, or leaves it untouched. The duality of Part V — the $\mathbb{PT}^+/\mathbb{PT}^-$ division as the frequency splitting — is the mathematical seam along which the two programs meet.

Three lines of the table deserve prose. *The seam:* on positive frequency the two programs state the same mathematics, and the only difference is the direction of explanation — which is exactly what a duality should look like from its two ends. *The massive row:* what the twistor ideal experiences as breaking (mass ruins conformal invariance) the present framework experiences as structure (mass lives on the scale fiber, and the clock-level derivation never needed the conformal group) — so the two programs are strong on complementary sectors, and the bridge says their strong sectors agree where they overlap. *The googly row:* honesty requires the table to show at least one hurdle that neither side has cleared; the nonlinear constructions are it, and objection O-P4-1 records the conjecture and the burden.

46 Programmatic verification

Every symbolic claim of Parts IV–VI has been verified programmatically in the parent program's verification suite, in addition to the proofs above: the circle and clock structures square to $-\mathbf{1}$ and agree exactly on Feynman–Stückelberg-paired modes (symbolically, including generic superpositions); the Jarlskog invariant of the CKM matrix is computed with zero complex arithmetic — real rotation blocks, conjugation as block transpose — matching the standard computation to machine precision, with rephasing invariance realized as dial re-zeroing invariance; the ground state of an arbitrarily coupled gapped quadratic lattice carries the polar-phase J of its interacting generator to machine precision, and Møller operators transport J across a defect-scattering event at the 10^{-3} level; the Hardy extension of a positive-frequency signal satisfies the Cauchy–Riemann equation numerically, recovers its boundary value, and exhibits the $e^{+|\omega|s}$ blow-up on the negative-frequency component; and the infrared overlap (35) vanishes as a power of the cutoff over twelve decades while the coherent shift leaves J invariant exactly; and the door theorem carries four dedicated checks — the cloud norm's logarithmic slope against the analytic coefficient of Lemma 38.3, the sector-overlap power

Table 5: Structural comparison of the twistor program and GG-Theory, aspect by aspect, with the seam (the holomorphy bridge, T-P4-6/7) marked. Twistor-column entries follow the program's own literature; the last column names where each GG-6 statement lives. Twelve aspects, from the origin of the complex numbers to experimental exposure. Where the twistor program names a hurdle of its own the row says so, and where this paper leaves a question open on both sides it says that too.

Aspect	Twistor theory	GG-Theory	Where
Origin of $\mathbb{C}$	Primitive: holomorphy is constitutive of nature	Derived: i = polar phase of the clock, geometrized by the circle; holomorphy = positivity at symmetry	T-P4-1–7
Positive frequency	The division $\mathbb{PT}^+/\mathbb{PT}^-$; central and exact	The same division, identified with the causal-positive splitting of the derived J — the seam	T-P4-7
Massless fields	Native habitat: $H^1(\mathbb{PT}^{\pm}, \mathcal{O}(-2h-2))$	Explained as the conformal-enhancement case; the free massless sector closed	T-P4-7
Massive fields	Long-standing hurdle (multi-twistor descriptions; conformal breaking)	Natural: the clock-level J needs no conformal symmetry; mass enters via the scale fiber and the gap	T-P4-2, T-P4-8
Why masslessness is special	A fact of the formalism	A theorem: only conformal fields complexify every timelike direction at once	T-P4-7(iii)
Interactions	Renaissance via amplitudes and the twistor string (Witten, 2004); full interacting theory open	Conditional completion by the asymptotic route (gap + completeness); infrared door proved at the model level	T-P4-8, T-P4-9
Both helicities of gravity ("googly")	The program's named central hurdle (Penrose, 1976)	Not addressed here; the nonlinear constructions remain open on both sides (O-P4-1)	§44
Why four dimensions	Twistor geometry is 4D-specific; taken as given	Derived: the arena is forced to $4+1+1$ with a 4D boundary	Arisaka, 2026, A2-GG6
Quantum state reduction	Hoped-for connection to objective reduction; unrealized	Carried by a different mechanism (open-system projection, attractor descent), with a decidable experimental fork against OR	Arisaka, 2026, A8-QM
Time asymmetry	Motivating concern; not delivered by the formalism	Constitutional: the boundary law is a forward semigroup; the arrow is the scale fiber's	Arisaka, 2026, A5-GDOS
Conformal breaking (m, Λ)	External to the twistor ideal	Located: mass on the scale fiber; the residual dark-energy share in the closed inventory	Arisaka, 2026, A10-Cosmos
Experimental exposure	Indirect (via amplitudes)	Named falsifiers: shared-i tests, lattice-locked phases, neutrino nature	§48

against its quadrature value, the dressed ground state's J against the free J while the sector label diverges, and the conjugation (37) to machine floor. The suite accompanies the program's working records and is available, together with the source, through the program's site (www.arisaka-gg.org).

47 The Platonic concession

One boundary is drawn deliberately. Everything above concerns nature's use of complex numbers; none of it touches their standing as mathematics. That the complex field is algebraically closed, that complex analyticity is rigid and consequence-rich, that the shortest paths between real truths so often pass through $\mathbb{C}$ — these facts are exactly as remarkable after this preprint as before it, and a mathematical Platonist is entitled to keep their wonder intact. The claim here is narrower and, within its regime, final: nature never needed the invention. The cosmos carries a circle, a clock, and an orientation; the complex numbers are how a four-dimensional observer, blind to the circle and embedded in the clock, compresses what it cannot see. The chronology of Part I tells the same story in dates: the compressed format was invented three centuries before the thing compressed was found, and the felt magic of 1926 was that gap coming due.

48 Predictions for falsification

The account is exposed on real fronts. A network-Bell violation of complex quantum mechanics itself — or any demonstration that independent sources can carry *unshared* complex structures — would break the shared-source picture (T-P4-4). A measured drift of the flavor CP phases away from their lattice values would retire the holonomy account of CP violation (the parent program's falsifier, Arisaka, 2026, A9-SM). An established Dirac nature for the light neutrinos would retire the dial-free-door consequences downstream (Arisaka, 2026, A10-Cosmos). And within mathematics, the bridge is checkable line by line: its components are classical theorems, and the identification of the twistor division with the frequency splitting is either exact or wrong.

49 Open questions

Six items, in decreasing order of consequence. (1) *The full-QED completion of the infrared door* (§38): T-P4-9 closes the door at the rigorous-model level — exactly at the van Hove stratum, on imported theorem-grade hypotheses at Nelson/Pauli–Fierz; what remains is the top rung, a rigorous dressing theory for full relativistic QED (the Faddeev–Kulish program). This is an existence question about dressed states that the field itself owns, and on which clauses (ii) and (iii) of the door theorem wait verbatim. (2) *The nonlinear twistor constructions*: whether the nonlinear graviton and the Ward correspondence (Ward, 1977) are the integrable packaging of the same two sources (Mason and Woodhouse, 1996) — the conjecture recorded at O-P4-1, and the one place the bridge has not yet reached. (3) *The dynamics half of the bridge*: T-P4-7 sources the *division* of twistor space in causal positivity; the twistor program's further magic — that holomorphy *generates* dynamics, the contour integral of an arbitrary holomorphic function solving the field equations automatically — is exhibited mechanically in Course B (homogeneity and cohomology) but not yet *sourced*. Deriving the holomorphic calculus itself from the real data is the sharpest Penrose-specific refinement this account admits. (4) *The exact-Kerr Hadamard analysis and the full gravitating collapse case*: their model-level cores are closed in the parent treatment (Arisaka, 2026, A8-QM, entries R14–R15) — the rotating anchor moves to the asymptotic frame with superradiance as a pumping channel, and the collapse epoch factorizes into a universal thermal channel times a decaying transient — while the exact curved-space existence theory remains the field's own. (5) *The modular route*: whether Tomita–Takesaki modular flow supplies the clock structure intrinsically, tying the CPT operator to modular conjugation — the long-horizon formulation. (6) *The typicality target upstream*: the parent program's measurement-theory question — whether deterministic basin-capture frequencies coincide with the Born measure — on which nothing in this paper depends. It has since been closed at the relaxation level in the parent treatment (Arisaka, 2026, A8-QM): the Born density is the unique conserved real density, dial-blindness and record stability force the transport, and the boundary law's own dissipation — a forward semigroup anchored on Haar by the compact fiber — relaxes any admissible initial configuration to the Born measure, monotonically and with no statistical postulate remaining; what remains are rate and representation estimates, and a relic-non-equilibrium falsifier. (The finite-volume zero-mode corner is not on this list, because it closes by Stone–von Neumann: finitely many modes admit one representation up to unitary equivalence, so the zero-mode J-choice is convention, not structure — the harmless finite-dimensional shadow of the infrared sectors.) None of these threatens the theorems as stated; each bounds the regime the theorems own.

50 The completed program and future prospects

Within the twenty-two-paper cohort, this paper completes the complex-variables arc: the real-bulk series RB-1–RB-12 of Arisaka, 2026, A8-QM supplies the statements, the Standard-Model and cosmological consequences are carried by Arisaka, 2026, A9-SM and Arisaka, 2026, A10-Cosmos, and the present paper is the standalone account addressed outward. The natural next steps are the full-QED completion of the infrared door, the nonlinear-twistor conjecture, and — on the expository side — the trimmed account for a broader venue once the full preprint has weathered its referees.

51 Summary of the Discussion

One derived complex structure, two real sources, three symmetry levels, twelve compared aspects, and a comparison with the two great alternative roads that respects both: the case, in numbers. What is proved is proved at the kinematic quasi-free level with a conditional interacting completion; what is open is five named items; and what is conceded — the mathematics of $\mathbb{C}$ itself — is conceded once and permanently.

Close of Part VII, in plain terms. The case has been weighed: significant to five constituencies, unique in the conjunction it fills, compared without polemic to the twistor and string roads, verified mechanically, fenced by named falsifiers, and bounded by five open questions — with the Platonic remainder conceded once and permanently. The Discussion's register has been deliberate: claims graded, credits paid, exposure listed. The next Part sets the ledger aside and speaks plainly — what was established, what it would mean, and why a half-century-old question deserved this long an answer.

Part VIII
Conclusions

What the paper has established; the objective limitations; a personal perspective; the vision of a century in which number systems are read as geometry; and what either verdict would mean

52 Part VIII in plain English: what has been established, and at what cost

The claim, stated plainly: nature's complex numbers are not primitive. They are the compression of two real structures — a compact phase circle and a causally oriented clock with a positivity clause — and holomorphy is what those two structures look like at sufficient symmetry. The gauge phase and the quantum i coincide for charged fields, which is why a century of quantum electrodynamics could not tell them apart.

The limitations are stated without cushioning, because they are the honest measure of the claim. The theorems are kinematic and quasi-free; the interacting extension is conditional on the standard scattering hypotheses; the infrared corner is closed at the model level and not for full quantum electrodynamics; the nonlinear twistor constructions are untouched by the bridge as proved; and the selection theorem leans on the staticity of the arena, whose gravitating generalization is inherited rather than owned. Each of the four carries a named next task.

What follows is a personal section and a longer view — the suggestion that a century which learned to read geometry as physics might come to read number systems the same way — and both are labeled as what they are rather than presented as results. The Part closes on what either verdict would mean, because a claim worth making is one that would cost something to be wrong about.

53 What the paper has established

Strip the technical casing and four sentences remain. The complex phase of every charged particle is a real angle on a real circle — a sixth dimension too small and too non-spatial ever to be looked around — and everything quantized about charge is that circle's compactness. The i of quantum mechanics is the phase of time itself: the unique way of marking "a quarter cycle ahead" that causality and positive energy permit, real in its construction, identical for every mode of every field. Antiparticles exist because those two facts must agree wherever charge lives. And holomorphy — the property Penrose called the magic of complex numbers, up to and including twistor space — is what the second fact looks like when a massless field's full symmetry is allowed to act on it: the division of twistor space into its two halves *is* the division of histories into forward and backward turning.

Behind the four sentences stand nine theorems, three courses, six worked examples, and a verification suite; but the four sentences are the paper, and it is worth saying why they carry philosophical weight beyond their proofs. Each replaces a brute fact with a consistency condition: charge quantization was a brute integer, and is now compactness; the antiparticle was a brute doubling, and is now a handshake; antiunitary time reversal was a brute table entry, and is now the only reversal that touches what i is made of; twistor holomorphy was a brute miracle, and is now positivity at full symmetry. Explanation in physics has always meant this trade — fewer brute facts, more consistency conditions — and the complex numbers, the oldest brute fact in the quantum formalism, have now been traded.

54 The objective limitations and the next technical tasks

Stated without cushioning: the theorems are kinematic and quasi-free, with the interacting extension conditional on the standard scattering hypotheses; the infrared corner of interacting massless theories is closed at the rigorous-model level (T-P4-9: sectors are ports, the derived J sector-blind), with the full-QED completion named; the nonlinear twistor constructions are untouched by the bridge as proved; and the selection theorem leans on the arena's staticity, whose gravitating generalization is inherited, not owned. Each limitation has a named next task, and none of the four is softer than the frontier every neighboring program faces on the same ground.

Each limitation carries its named next task, and they are worth listing as a program rather than a disclaimer. The interacting completion now carries its model-level infrared theorem (T-P4-9, Nelson/Pauli–Fierz); what it wants next is a rigorous full-QED dressing theory. The nonlinear twistor frontier wants a single worked case: one self-dual solution exhibited as integrable packaging of the two sources, or a proof that it cannot be. The rotating anchor wants the asymptotic-frame formulation written out against the Kay–Wald obstruction. And the staticity dependence wants the gravitating generalization stated, even conditionally. None of these is decorative: each is a place where this account could break, and the paper's wager is that it will not. A program that names its own failure modes has, at minimum, earned the right to be tested rather than dismissed.

55 A personal perspective

I have spent fifty years on the experimental front line of particle physics, and the number system in the theory columns of my logbooks never once appeared in the data columns. Detectors report times, counts, angles, energies: real numbers, every one. The amplitudes were complex; the entries never were. For most of those decades I filed this under bookkeeping and moved on — until the question was put to me in Penrose's voice, as a challenge that a geometric program ought to meet head-on: if your bulk is real, where does the i live? This paper is the long answer. The short answer, which I offer to any experimentalist who has wondered the same thing between shifts, is that the i lives exactly where the data said it did: nowhere. What lives are a circle we cannot look around and a clock we cannot run backward, and the complex plane is the shorthand four-dimensional beings invented for the view.

One more sentence of the personal register, because students ask. The strangest part of writing this paper was not the mathematics; it was the repeated experience of finding the key already cut. Hamilton had the pairs; Gabor had the signal; Penrose had the division; Stueckelberg had the real space; the experimentalists had the shared i. Nobody was wrong, and almost nothing here is technically new at the component level — what was missing was only the insistence, from a program built on real geometry, that the components be laid on one table and asked whether they were a single object. They were. Fifty years of logging real numbers from detectors made me stubborn enough to ask; the reader may judge whether the stubbornness paid.

56 The vision: reading number systems as geometry

The geometrization of physics has climbed a familiar ladder: space (Euclid to Riemann), time (Minkowski), gravity (Einstein), gauge force (Weyl to Yang–Mills), scale (Wilson to the holographic renormalization group). The parent program's wager is that the ladder ends only when everything the dynamics cannot avoid has been made geometry. This paper proposes that the ladder's next rung was hiding in the number system itself: that $\mathbb{C}$ — the one piece of apparatus every physics course installs before the physics begins — is also a geometrization waiting to be recognized, the circle and the clock compressed into a symbol. If that is right, then a century from now the phrase "complex quantum amplitude" may read the way "luminiferous aether" reads to us: not wrong, but a name for structure seen before it was located. The mathematics will not change; the understanding of what it was about will.

57 The trailhead, not the summit: a final reflection

A reader should leave this paper with proportions intact. What stands proved here is a set of identifications — exact, checkable, and (we believe) new — between structures that were each already understood alone. What stands open is the harder half of every such program: the full-QED infrared, the nonlinear, the gravitating. And what stands entirely outside physics is the wonder that started the question: the uncanny excellence of complex analysis as mathematics. We have argued that nature never needed the invention; we have not argued, and would not, that the invention is less than miraculous. The trail continues uphill from here, in at least five marked directions; this paper is the trailhead sign, not the summit cairn.

And the vision has a pedagogy. If this account settles, the curriculum inverts: a student would first meet real oscillation pairs and the dial, then be *shown* the compression — here is the plane, here is the quarter-turn, here is why one symbol carries both — and only then be handed $e^{i\theta}$, the way phasors are handed to engineers, as owned shorthand rather than as metaphysics. The C, P, T table would be taught as orientation bookkeeping in an afternoon. Twistor geometry would enter as the conformal completion of the frequency dial rather than as an isolated miracle. None of this changes a calculation; all of it changes what a student believes physics is made of — and a century of students who never mistake the portrait for the sitter is the quiet, compounding payoff of getting a foundation right.

58 What it would mean: mathematics, nature, and the observer

If the account survives its referees — the human ones, and the experimental ones listed under falsification — then something gentle but permanent happens to the relationship between mathematics and physics. The question "why is mathematics so unreasonably effective?" loses one of its sharpest examples and gains a case study: a piece of mathematics invented for pure algebra turned out to be effective because it was, unknowingly, a portrait — of a circle nature carries at every point and a clock nature carries along every worldline. The effectiveness was not unreasonable; it was unrecognized. And the moral runs in both directions: toward nature, whose inventory proves leaner and more real than its notation suggested; and toward us, the four-dimensional observers, whose compressions are so good that it took four centuries to notice they were compressions. Physics is the art of telling the portrait from the sitter. This paper has tried to perform that art on the oldest portrait in the gallery — and to hand the twistor school, with respect and in its own language, the sitter its geometry has been drawing all along.

59 The last word: why complex numbers

Why complex numbers? Because a real circle, too small and too non-spatial ever to be looked around, turns the harmonic analysis of charge into complex exponentials; because a real causal order, with energy positivity, turns every history into a positive-frequency half that is secretly the boundary of a holomorphic function; because the two agree wherever charge lives, at the price of antiparticles; and because a four-dimensional observer, meeting the compressed format everywhere and its sources nowhere, concluded that nature runs on a number system invented for cubics. It does not. Nature runs on angles, orderings, and orientations — all of them real — and the complex numbers are the shadow those cast on the boundary where we live. The half of the magic that was physics is hereby returned to geometry; the half that is mathematics is returned, with gratitude, to the mathematicians — and to the man who insisted for fifty years that the question deserved an answer.

Close of Part VIII, in plain terms. The summit, restated once without apparatus: nature runs on angles, orderings, and orientations, all real; the complex numbers are the shorthand four-dimensional observers invented for the view; and the oldest magic in the quantum formalism is geography. For the reader who wants to own the machinery rather than admire it, the remaining Parts are the workshop: three courses with problem sets, the reference dictionaries, an annotated guide to the literature — and then the forty-two sharpest attacks we could assemble, each answered, because a claim of this size should leave with its defense attached.

Acknowledgments

This work was developed through extensive iterative collaboration with three large language models: ChatGPT (OpenAI) for exploratory drafting, narrative development, and broad-scope ideation; Perplexity for literature synthesis, citation verification, and recent experimental cross-referencing; and Claude (Anthropic) for structural organization, mathematical consistency checking, LaTeX typesetting, and final manuscript preparation. The sequential refinement of the GG-Theory framework (GG-Constitution / GG-6 / BI-6 / GDOS / Gi-4 Pillars / Qi-4 Pillars) across hundreds of conversation sessions with these three systems was essential to the coherence of the present paper.

The author thanks Aaron Blaisdell and Hirotaka Iwasaki for stimulating discussions on the program's interdisciplinary scope and for critical feedback on the accessibility of the presentation. The author also thanks hundreds of collaborators in the Elegant Mind Club over the past decade, including Javier Carmona, Elizabeth Milles, Mingda He, Syed Bashir Hydari, Brian Ta, Chris Dao, Umaima Afifa, Isabella Bestanoby, Caominh Le, and Natsuko Yamaguchi.

All scientific concepts (especially the foundation of the GG-Constitution, GGEP/GG-6/BI-6, and OGGEP/GDOS/Gi-4/Qi-4 Pillars), theoretical judgments, physical interpretations, and any remaining errors are the sole responsibility of the author.

Part IX
Technical Appendices

The reference dictionaries and a pocket course — from the analytic signal to the Penrose transform

60 Summary of abbreviations

GG-6 (geometry–gauge); OGGEP (operational geometry–gauge equivalence principle, the dictionary axiom AX-3); GDOS (geometric dynamics of open systems, the boundary law); GRIP (the gauge-runtime invariance pipeline, the actualization layer); F–S (Feynman–Stückelberg); CR (Cauchy–Riemann); $\mathbb{PT}$, $\mathbb{PT}^{\pm}$, $\mathbb{PN}$ (projective twistor space, its halves, its null hypersurface); H^2 (Hardy space); T-P4-n (this paper's theorems); RB-n (the real-bulk series of Arisaka, 2026, A8-QM).

61 Glossary of key symbols

S^1_θ: the compact phase circle (sixth dimension). I_r: the scale interval (fifth dimension). J: a complex structure (Definition 10.2); $J_Q = \partial_\theta/Q$ the circle's, J_A the clock's polar phase, coinciding on shell (T-P4-3). H: the Hilbert transform (7). σ, s: the symplectic (bosonic) and Euclidean (fermionic) forms. $Z^\alpha = (\omega^A, \pi_{A'})$: a twistor; $s(Z)$ the helicity form (9). $\mathcal{T}^+$: the forward tube (26). (a, b): the real pair of a weight-n harmonic (20).

62 A pocket course: from the analytic signal to the Penrose transform

The bridge of Part V can be walked in five steps with a pencil; this appendix is the walk, condensed, with every formula on one page per step.

Step 1 (one mode). Take $f = \cos\omega t$. Form $g = \mathsf{H}f = \sin\omega t$; check $\mathsf{H}g = -f$. Define $F = \frac{1}{2}(f + ig) = \frac{1}{2}e^{i\omega t}$ and extend $t \mapsto t + is$: decay upstairs, growth downstairs. You have verified T-P4-6 on one mode and met the derived J.

Step 2 (all modes). Superpose. The multiplier form (7) makes H frequency-diagonal, so Step 1 holds mode by mode; Paley–Wiener (Theorem 13.3) upgrades the picture to the exact statement: one-sided support = Hardy class.

Step 3 (all clocks). Replace t by every timelike direction: a positive-frequency field decays under $x \mapsto x + iy$ for every future-timelike y, which is holomorphy in the forward tube (26). This is where Lorentz invariance enters and where massive fields stop.

Step 4 (light's symmetry). For a massless field the conformal group acts; the tube compactifies, and its geometry is exactly $\mathbb{PT}^+$: check on the incidence relation (8) that complexifying x into the forward tube moves the line L_x into the region $s(Z) > 0$. This is the content of T-P4-7(iii) in coordinates.

Step 5 (the transform). Verify the contour integral (28) on the elementary state of Worked example II: two poles, one contour, one rational solution of the wave equation — and observe that keeping the poles separable is exactly keeping the state on one side of $\mathbb{PN}$. Frequency sign has become pole placement; you have crossed the bridge on foot.

63 Course A: from the Hilbert transform to holomorphy

This appendix is a self-contained graduate course on the analytic core of Parts IV–V: six lectures, each with a from-scratch derivation and a worked example. A student who completes it owns every analytic tool the paper uses.

63.1 Lecture A.1: the transform, three ways

The Hilbert transform admits three equivalent definitions, and each carries a different intuition. *As a principal-value integral*, (7): a convolution against $1/\pi t$, the odd kernel that weighs the past against the future symmetrically — manifestly real, manifestly non-local. *As a Fourier multiplier*, $-i\,\mathrm{sgn}(\omega)$: frequency by frequency, a quarter-period advance — manifestly $J^2 = -\mathbf{1}$ away from $\omega = 0$. *As the conjugate-function operator*: the map sending the boundary values of the real part of a holomorphic function to the boundary values of its imaginary part — manifestly the bridge's germ. Equivalence of the three is the content of Lecture A.2.

Worked derivation: a classical pair. Let $f(t) = 1/(1 + t^2)$, the Poisson kernel at height one. Close the principal-value contour in the upper half plane around the pole at $t' = i$:

$$(\mathsf{H}f)(t) = \text{p.v.}\,\frac{1}{\pi}\int \frac{dt'}{(1 + t'^2)(t - t')} = \frac{t}{1 + t^2}, \tag{38}$$

the conjugate Poisson kernel: the transform rotates the "cosine-like" bump into its "sine-like" partner, exactly as it rotates $\cos \mapsto \sin$ mode by mode. Check the involution: applying (38) again returns $-1/(1+t^2)$, i.e. $\mathsf{H}^2 = -\mathbf{1}$ on this pair. Observe what the pair sums to: $f + i\mathsf{H}f = 1/(1-it) = i/(t+i)$ — a function with no singularity for $\operatorname{Im} t > 0$. Holomorphy has already appeared, uninvited.

Problems.

1. Compute H of the Gaussian-windowed tone $e^{-t^2/2}\cos\omega_0 t$ in the large-ω_0 limit, and show it approaches $e^{-t^2/2}\sin\omega_0 t$: the transform sees the carrier, not the envelope. *(Hint: multiplier form; the window's spectrum is narrow.)*

2. Show from the multiplier form that H commutes with translations and dilations of t but anticommutes with the reflection $t \mapsto -t$. Interpret the anticommutation in the language of T-P4-5. *(Answer: reflection reverses the clock's orientation, so it must reverse J.)*

3. Verify $\mathsf{H}^2 = -\mathbf{1}$ directly on the pair (38) by one more contour integral.

63.2 Lecture A.2: Hardy space and the Paley–Wiener theorem

Following the classical treatment (Titchmarsh, 1948), define the Poisson and conjugate-Poisson kernels at height $s > 0$, $P_s(t) = s/\pi(t^2 + s^2)$ and $Q_s(t) = t/\pi(t^2 + s^2)$, and for real $f \in L^2$ set $F(t + is) = (P_s * f)(t) + i(Q_s * f)(t)$. Three facts, each a short computation: (i) F is holomorphic in the upper half plane ($P_s + iQ_s = i/\pi(t + is)$ is, and convolution preserves it); (ii) as $s \downarrow 0$, $P_s * f \to f$ and $Q_s * f \to \mathsf{H}f$ in L^2 — so the transform *is* the boundary conjugate function; (iii) on the Fourier side, $\widehat{P_s * f} = e^{-s|\omega|}\hat{f}$ and $\widehat{F(\cdot + is)} = 2e^{-s\omega}\hat{f}\,\chi_{\omega>0}$ — the extension keeps only the positive frequencies and damps them. Reading (iii) in both directions is the Paley–Wiener theorem (Theorem 13.3): one-sided support iff Hardy-class extension; and the splitting projector $\frac{1}{2}(\mathbf{1} + i\mathsf{H})$ is the boundary shadow of "extend upward and keep what survives."

Worked example. Take $f = \cos\omega_0 t + \frac{1}{3}\cos 3\omega_0 t$ (a two-mode chord). Then $F(t + is) = \frac{1}{2}e^{i\omega_0(t+is)} + \frac{1}{6}e^{3i\omega_0(t+is)}$: each overtone decays at its own rate $e^{-\omega s}$, the chord's envelope and instantaneous phase are read off $|F|$ and $\arg F$, and the higher-frequency component dies faster with height — the upper half plane acts as a frequency microscope, which is the analyst's picture of why one-sided regularity encodes orientation.

Problems.

1. Prove that $\frac{1}{2}(\mathbf{1} + i\mathsf{H})$ is idempotent on zero-mean signals and identify its kernel and range. *(Answer: negative- and positive-frequency subspaces respectively.)*

2. For $f = \cos\omega_0 t$, compute $\int |F(t+is)|^2 dt$ as a function of height s and confirm the Hardy bound is saturated by monotone decay $e^{-2\omega_0 s}$.

3. A signal has spectrum supported in $[\omega_1, \omega_2]$, $0 < \omega_1 < \omega_2$. Show its extension is entire in a strip and estimate the width. *(Hint: two-sided decay.)*

63.3 Lecture A.3: the polar phase, computed

Lemma 11.1 in coordinates. Take the smallest non-trivial case: two unit-mass oscillators with stiffness matrix $K = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ (coupled springs; eigenfrequencies $\omega_\pm = 1, \sqrt{3}$). The generator on (φ, π) data is $A = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ -K & 0 \end{smallmatrix}\right)$, and the polar phase evaluates in closed form:

$$J_A \;=\; A\,(-A^2)^{-1/2} \;=\; \begin{pmatrix} 0 & K^{-1/2} \\ -K^{1/2} & 0 \end{pmatrix}, \qquad K^{1/2} = \tfrac{1}{2}\begin{pmatrix} \sqrt{3}+1 & 1-\sqrt{3} \\ 1-\sqrt{3} & \sqrt{3}+1 \end{pmatrix}. \tag{39}$$

Check by block multiplication: $J_A^2 = -\mathbf{1}$; symplectic compatibility from $K^\mathsf{T} = K$; and $[J_A, e^{tA}] = 0$ since both are functions of A. Notice what the off-diagonal blocks say physically: "multiply by i" mixes position into momentum with the matrix $K^{-1/2}$ — each normal mode advanced a quarter of *its own* period. The exact ground state of this pair has covariance $\Gamma = \frac{1}{2}\,\mathrm{diag}(K^{-1/2}, K^{1/2})$, and $2\Gamma\sigma = J_A$ on the nose: the vacuum knows the polar phase.

Problems.

1. Repeat the closed-form computation (39) for the antisymmetric normal-mode pair $K = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ and check that only the eigenvector labels move. *(Moral: the polar phase depends on the spectral planes, not on your basis.)*

2. Show that for any $\alpha > 0$, $J_{(\alpha A)} = J_A$: the polar phase is scale-free. Which physical statement in §26 does this prove? *(Answer: one i for all modes regardless of frequency.)*

3. Let A have a zero mode. Show no dynamics-commuting complex structure exists on the full space, and identify the obstruction with O4 of the main text.

63.4 Lecture A.4: from J to the Hilbert space

With (V, σ, J) in hand, the one-particle inner product (3) turns the real solution space into a complex Hilbert space with $i = J$. On the oscillator pair of A.3: the normal-mode data $(q_\pm, p_\pm)$ give $\langle f, f\rangle = \sum_\pm \frac{1}{2}(\omega_\pm q_\pm^2 + p_\pm^2/\omega_\pm)$ — the classical energy weighted by $1/\omega$ mode by mode, i.e. the *action*: the quantum norm is the classical action in disguise, which is why $\hbar$ counts quanta. Creation and annihilation appear as the $(\mp i)$-eigenvectors of J, $a_\pm \propto \omega_\pm^{1/2} q_\pm + i\,\omega_\pm^{-1/2} p_\pm$: the textbook combination, here *derived* as "the polar phase's own coordinates" rather than guessed.

Problems.

1. Verify on the oscillator pair that $\langle f, g\rangle$ of (3) is sesquilinear with respect to J: $\langle Jf, g\rangle = -i\langle f, g\rangle$ under the identification $i \equiv J$.

2. Show that the classical time-average of the energy in a single mode equals ω times the one-particle norm-squared: the norm counts quanta. *(This is why $\hbar$ appears where it does.)*

3. Construct the $(\pm i)$-eigenvectors of J_A for the coupled pair of A.3 explicitly and confirm they diagonalize the Hamiltonian.

63.5 Lecture A.5: fields, the tube, and the massive/massless contrast

Promote A.3–A.4 to the Klein–Gordon field: V = real solutions, σ = (1), A = time translation, $-A^2 = m^2 - \Delta$ — gapped for $m > 0$, gapless for $m = 0$ but with $\omega = |k| > 0$ almost everywhere. The polar phase is multiplication by $-i\,\mathrm{sgn}$ on frequencies, i.e. the field Hilbert transform; positive-frequency solutions extend to the forward tube (26) because $e^{ik\cdot x}$ with $k \in \bar{V}^+$ decays under $x \mapsto x + iy$, $y \in V^+$ $(k \cdot y > 0)$. The contrast that matters for Course B: for $m > 0$ the extension exists but the tube's conformal compactification is unavailable (mass fixes a scale); for $m = 0$ the conformal group acts, and the tube's geometry acquires the twistor description. Same theorem, two destinies.

Problems.

1. For the massless field in a periodic box, list the modes on which J is undefined and count them. *(Answer: the single constant mode; measure zero in infinite volume.)*

2. Show $e^{ik\cdot x}$ with k future-null decays in the forward tube but not in any wider domain: the tube is exactly the shared decay region.

3. Where does the argument use the mass gap? Re-run A.5 for $m > 0$ and identify the one step that fails at $m = 0$ in finite volume.

63.6 Lecture A.6: where the analysis can fail, quantified

Three failure modes, each with its number. *Zero modes:* sgn(0) is undefined; in finite volume the constant mode of a massless field is a genuine degree of freedom needing separate (classical/superselected) treatment. *Non-stationarity:* with a time-dependent generator the splitting at $t = -\infty$ and $t = +\infty$ differ by a Bogoliubov map; in the one-mode case $\left(\begin{smallmatrix}\cosh r & \sinh r\\ \sinh r & \cosh r\end{smallmatrix}\right)$, the "out" vacuum contains $\bar{n} = \sinh^2 r$ "in" quanta — particle creation as the measured mismatch of two polar phases, the physics behind Worked example IV. *The interacting infrared:* the overlap exponent (35) evaluates to $(\epsilon/\Lambda)^{|\Delta c|^2/2}$; for $|\Delta c|^2 = 8$ and twelve decades of cutoff removal the overlap falls by 10^{-48} — sectors, for every practical and every principled purpose. Each failure mode bounds the theorems of the main text exactly where Part VI said it would.

Problems.

1. For the Bogoliubov block with $r = 1$, compute $\bar{n}$, the two-mode entanglement entropy $S = (\bar{n}{+}1)\ln(\bar{n}{+}1) - \bar{n}\ln\bar{n}$, and the KMS temperature of the reduced state.

2. Evaluate the infrared overlap (35) for $|\Delta c|^2 = 2$ and cutoffs $\epsilon/\Lambda = 10^{-3}, 10^{-6}, 10^{-9}$; plot mentally, conclude sectorially.

3. Derive (37): show that $W(d)^\dagger\,(\omega a^* a + g(a + a^*))\,W(d) = \omega a^* a - g^2/\omega$ at $d = -g/\omega$, and conclude with Lemma 38.1 that the sector's polar-phase J is the free one. Then compute the logarithmic slope of $\|f_v\|^2$ from (36) at $v = 0.4$ and check it against $2\pi g^2/(1 - v^2)$.

4. Show that a coherent shift composed with a Bogoliubov map is again affine-plus-symplectic, and that only the Bogoliubov factor can move J. *(This separates Worked examples IV and the cloud lemma cleanly.)*

64 Course B: twistor geometry from scratch

Seven lectures from two-component spinors to the frequency theorem, assuming only special relativity and linear algebra. The destination is the one derivation this paper most wants its reader to own: that a spacetime point pushed into the forward tube lands its twistor line in $\mathbb{PT}^+$ — the bridge, computed.

64.1 Lecture B.1: two-component spinors

Represent a spacetime point by the Hermitian matrix

$$x^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}, \qquad \det x = \tfrac{1}{2}(t^2 - x^2 - y^2 - z^2), \tag{40}$$

so Lorentz transformations act as $x \mapsto \Lambda x \Lambda^\dagger$ with $\Lambda \in SL(2,\mathbb{C})$, and null vectors are exactly the rank-one Hermitian matrices $x^{AA'} = \kappa^A \bar{\kappa}^{A'}$: a null direction *is* a spinor up to phase. This factorization is the engine of everything below — light is spinor-valued, which is why light's geometry will complexify so gracefully.

Problems.

1. Verify $\det x = \frac{1}{2}(t^2 - x^2 - y^2 - z^2)$ from (40) and show $x \mapsto \Lambda x \Lambda^\dagger$ preserves it: $SL(2,\mathbb{C})$ double-covers the Lorentz group.
2. Write the null vector $(1,0,0,1)$ in the form $\kappa\bar{\kappa}$ and exhibit the phase freedom $\kappa \mapsto e^{i\phi}\kappa$. What does this phase become in Lecture B.3? *(Answer: the little-group direction the helicity form measures.)*
3. Show that two spinors κ, μ with $\kappa_A \mu^A \neq 0$ span every spinor: the two-component calculus needs only antisymmetric contraction.

64.2 Lecture B.2: twistors and incidence

A twistor is a pair $Z^\alpha = (\omega^A, \pi_{A'})$; a point x and a twistor Z are *incident* when (8) holds, $\omega = ix\pi$. Fix Z with $\pi \neq 0$: the incident real points form a null geodesic with direction $\pi^A \bar{\pi}^{A'}$ — a light ray, with ω recording its moment arm. Fix x: the incident twistors form a two-complex-dimensional subspace, the projective line $L_x \subset \mathbb{PT}$. Points are lines; rays are points; two points are null-separated iff their lines meet. The dictionary inverts the habits of a lifetime, which is precisely its power: what is local in $\mathbb{PT}$ is global in spacetime, and vice versa.

Problems.

1. From (8), derive that the incident real points of a fixed Z (with $\pi \neq 0$) form the stated null ray, and compute its direction. *(Hint: difference of two solutions.)*
2. Show two points x, y are null-separated iff $L_x \cap L_y \neq \emptyset$, and express their separation vector via the intersection twistor.
3. What happens to L_x as $x \to \infty$ along a timelike direction? *(Answer: it approaches the line of the vertex twistor at infinity — compactification made visible.)*

64.3 Lecture B.3: the helicity form, and what $s(Z)$ measures

The Hermitian form (9) is the unique $SU(2,2)$-invariant on twistor space, and its physical name is earned: in the conventions of Penrose and Rindler (1986) (§6.1), a one-particle twistor carries momentum and angular momentum

$$P_{AA'} = \bar{\pi}_A\,\pi_{A'}, \qquad M^{AB} = i\,\omega^{(A}\bar{\pi}^{B)}, \qquad M^{A'B'} = -\,i\,\bar{\omega}^{(A'}\pi^{B')}, \tag{41}$$

and the Pauli–Lubański (helicity) invariant built from them evaluates to exactly $s(Z)$ (Penrose and Rindler, 1986). On incident pairs the form vanishes: substituting $\omega = ix\pi$ with x Hermitian gives

$$s(Z) \;=\; \tfrac{1}{2}\big(ix^{AA'}\pi_{A'}\bar{\pi}_A - ix^{AA'}\bar{\pi}_A\pi_{A'}\big) \;=\; 0, \tag{42}$$

so real spacetime points live entirely on the null hypersurface $\mathbb{PN}$ — the "real slice" of twistor space is where helicity-neutral incidence happens, and the two open halves $\mathbb{PT}^{\pm}$ are, so far, unclaimed territory.

Problems.

1. Verify (42) in full index detail, keeping track of which reality property of x is used where.
2. Compute $s(Z)$ for Z incident with the complex point $x + i\epsilon e_0$ (e_0 the unit time direction, $\epsilon > 0$ small) to first order in ϵ, and check the sign against Lecture B.4.
3. Using the momentum expression $P_{AA'} = \bar{\pi}_A\pi_{A'}$, show a rescaling $\pi \mapsto \lambda\pi$ rescales the null momentum: projective twistors carry rays' worth of momenta, not momenta.

64.4 Lecture B.4: the derivation this course exists for

Now push the point into the forward tube: $x \mapsto x - iy$ with y future-timelike Hermitian ($y = \lambda\lambda^{\dagger}$-decomposable with positive trace). The incident twistors of the complexified point have $\omega = i(x - iy)\pi$, and the helicity form no longer vanishes:

$$s(Z) \;=\; \tfrac{1}{2}\big(\omega^A\bar{\pi}_A + \bar{\omega}^{A'}\pi_{A'}\big) \;=\; \tfrac{1}{2}\big(i(x-iy) - i(x+iy)\big)^{AA'}\pi_{A'}\bar{\pi}_A \;=\; y^{AA'}\,\pi_{A'}\bar{\pi}_A \;>\; 0, \tag{43}$$

strictly, because $y^{AA'}\pi_{A'}\bar{\pi}_A$ is the Minkowski pairing of the future-timelike y with the future-null direction $\pi\bar{\pi}$. *The entire line L_{x-iy} sits in $\mathbb{PT}^+$.* One line of spinor algebra, and the bridge is crossed: forward-tube analyticity (Course A's destination) is exactly "twistor lines in the positive half" — T-P4-7(iii), by hand. Push x into the *backward* tube and the same computation lands the line in $\mathbb{PT}^-$: the two halves of twistor space are the two frequency orientations, made into geography.

Problems.

1. Re-run the computation (43) for y *past*-timelike and for y spacelike; identify where each lands. *(Answer: $\mathbb{PT}^-$; and straddling both halves respectively.)*

2. Deduce from (43) alone that no line L_z of a complexified point can lie in $\mathbb{PN}$ unless z is real: the null hypersurface is exactly the real slice.

3. Combine B.4 with A.5 to give a two-line proof of T-P4-7(ii): a positive-frequency field restricted to a worldline is Hardy.

64.5 Lecture B.5: homogeneity, the contour integral, and the residue

Penrose's original transform: for helicity 0, (28), with f holomorphic of homogeneity -2. Execute it for the elementary state $f(Z) = [(A_\alpha Z^\alpha)(B_\beta Z^\beta)]^{-1}$. Parameterize the line L_x by $\pi_{A'} = (1, \zeta)$; the two factors become linear in ζ, with roots $\zeta_A(x), \zeta_B(x)$; the measure $\pi\, d\pi = d\zeta$; and the contour separating the roots picks up one residue:

$$\varphi(x) \;=\; \frac{1}{2\pi i}\oint \frac{d\zeta}{a(x)\,(\zeta-\zeta_A)(\zeta-\zeta_B)\,b(x)} \;=\; \frac{1}{\rho_A(x)\,\rho_B(x)}, \tag{44}$$

with $\rho_{A,B}$ the incidence factors — a rational, everywhere- defined-off-singularities solution of $\Box\varphi = 0$, produced by *contour topology alone*. Homogeneity -2 is what makes the integrand a well-defined one-form on the projective line (count: -2 from f, $+2$ from $\pi\, d\pi$); for helicity h the same count forces $-2h-2$, which is the exponent in (11). Frequency, meanwhile, is the *placement* of ζ_A, ζ_B: a positive-frequency state is one whose singularities can be kept off $\mathbb{PT}^+$, so that on every L_{x-iy} the contour exists — Lecture B.4 again, now running the transform.

Problems.

1. Carry out the residue computation (44) keeping all factors, for the specific dual twistors $A = (0, \iota)$, $B = (0, o)$ with ι, o a spinor basis, and identify the singularity surfaces of the resulting φ.

2. Run the homogeneity count for helicity $h = \pm 2$ and match the exponents to the linearized-graviton entries of (11).

3. Show that deforming the contour across one pole changes φ by a solution regular at that pole's incidence region — the germ of the coboundary freedom of B.6.

64.6 Lecture B.6: why cohomology, in one page

The contour formula hides an ambiguity that is actually a structure: f is not a function on $\mathbb{PT}^+$ but a representative defined on an overlap of patches, and adding to it anything holomorphic on a single patch changes nothing (deform the contour off the added term's singularity-free region and the integral of that piece vanishes). Data on overlaps, modulo single-patch data: that is exactly a Čech cohomology class in H^1, computed with a two-patch cover. The mysterious-sounding statement "massless fields are first cohomology" is therefore the pedestrian statement "twistor functions are contour-integrands, and only their patch-overlap content is physical" — the same logic by which only loop holonomies of a texture are physical (T-P4-5), transplanted to complex geometry. The rigorous form is Eastwood *et al.* (1981); the two-patch picture is enough for every use in this paper.

Problems.

1. With the two-patch cover $\{\zeta \neq \zeta_A\}$, $\{\zeta \neq \zeta_B\}$ of the line, exhibit the elementary state of B.5 as a Čech cocycle and one coboundary that leaves φ unchanged.

2. Explain, in the holonomy language of T-P4-5, why H^0 carries no massless field: single-patch data are re-zeroable. *(One sentence suffices.)*

3. For which helicities does homogeneity permit *polynomial* cocycles, and what fields do they give? *(Answer: the finite-type solutions; try $h = 0$, degree -2.)*

64.7 Lecture B.7: the dictionary, closed

Assemble the course. A massless field of helicity h is a class in $H^1(\mathbb{PT}^{\pm}, \mathcal{O}(-2h-2))$ (B.5–B.6); its frequency orientation is which half carries it (B.4); on any timelike worldline that orientation restricts to the Hardy splitting (Course A); and the generator of that splitting is the polar phase of the clock — the derived J of the main text, whose other face, on charged sectors, is the rotation generator of the phase circle. Read left to right, the dictionary derives twistor geography from causal positivity; read right to left, it equips the real bulk's two structures with the most powerful complex-geometric calculus ever built for massless fields. The direction of *explanation* is the main text's claim; the dictionary itself, either way round, is now yours.

Problems.

1. Trace one electron through the full dictionary: its dial (T-P4-1), its clock phase (T-P4-2), their pairing (T-P4-3) — and explain why, being massive, it stops at the tube level of the bridge.

2. State exactly which step of the course fails for the googly problem, and why the failure is about *nonlinearity* rather than about frequency. *(Answer: B.5–B.6 linearize; self-dual deformation does not.)*

3. Write a one-paragraph referee report on T-P4-7 from a twistor-school perspective: what must be checked, and where would you probe? *(Model answer: objection O-P4-1 and its response.)*

65 Course C: the two real sources, at lecture depth

Courses A and B built the analysis and the geometry; this course drills the paper's own theorems, at the same depth and with the same apparatus. Six lectures, each ending in problems; a student who finishes all three courses has reconstructed Parts IV–V unaided.

65.1 Lecture C.1: the circle sector, drilled

Work entirely on one weight-n sector (20). Write the circle action, the covariant derivative, and the holonomy in the real pair (a, b); verify by direct block computation that the horizontal lift acts as $\partial_\mu - inA_\mu$ on $\psi = a + ib$ and as its conjugate on $\bar{\psi}$; and prove T-P4-1(iii)'s uniqueness by finding the general solution of $RJ = JR$ for all rotations R — a two-parameter family — and imposing $J^2 = -\mathbf{1}$. The physical drill: compute the Aharonov–Bohm shift for a loop enclosing flux Φ as a pure dial rotation, with no complex number appearing until the final repackaging.

Problems.

1. Show a weight-n and a weight-m sector admit an invariant *bilinear* pairing iff $m = -n$, and identify it with the Dirac pairing between a field and its conjugate. *(This is charge conservation as pure representation theory.)*

2. An instrument couples to $\cos\theta$ (it reads the absolute dial). Show its interaction violates the fiber isometry and estimate what its existence would do to charge superselection.

3. Derive the de Broglie relation of the parent program's dictionary: a dial winding n times per fiber circuit and $p/\hbar$ times per unit length has total angle $n\theta + p \cdot x/\hbar$; identify both windings' conservation laws.

65.2 Lecture C.2: the clock's uniqueness, in full

Lemma 11.1 asserted uniqueness; here is the complete argument. Let J' commute with the dynamics and be positive-energy. Decompose the solution space into the spectral planes of A (nondegenerate case first); commutation forces J' to preserve each plane; on a plane, the commutant's solutions of $J'^2 = -\mathbf{1}$ are the two quarter-turns; positivity of $\sigma(f, J'f)$ selects the same orientation as J_A on every plane, because both signs are determined by the same inequality $\omega > 0$. Degenerate frequencies: the commutant enlarges to a unitary group on the degenerate block, but positivity still forces $J' = J_A$ on the block's real form — run the argument with a two-fold degeneracy to see the mechanism. Conclude: uniqueness is not delicacy; it is the rigidity of "one orientation per plane, one inequality for all planes."

Problems.

1. Carry out the degenerate-block argument explicitly for two modes of equal frequency.

2. Show that dropping positivity yields a $2^{\#\text{planes}}$-fold family of invariant complex structures, and compute the energy spectrum's signature for one wrong choice. *(Answer: a negative one-particle branch — the fermionic flipped-block demonstration of the parent program, bosonically.)*

3. Prove that any two positive-energy choices for *different* static frames of the same flat arena agree. *(Hint: they share the frequency planes of the full Poincaré representation.)*

65.3 Lecture C.3: the handshake, drilled

T-P4-3's computation (23), three more times: once with general phases (mode $\cos(n\theta - Et + \varphi)$ — the handshake is phase-blind); once for the antiparticle assignment (charge $-n$, positive frequency — agreement restored by the signed normalization); once for the forbidden assignment, isolating the sign. Then the discrete table as a drill: apply C, P, T, CP, CPT to the general mode and tabulate which operations flip the relative sign of J_Q and J_A — reproducing T-P4-5's table from raw substitutions.

Problems.

1. Show that demanding the handshake for *two* charges n_1, n_2 simultaneously forces the same frequency orientation for both: one clock serves every sector, which is T-P4-4(i) in miniature.
2. A hypothetical "anti-handshake" theory pairs $+n$ with negative frequency. Show it is unitarily equivalent to the standard one by circle reversal, and identify which discrete symmetry implements the equivalence.
3. Where exactly does the argument use compactness of the fiber? *(Answer: nowhere in the handshake itself — but integrality of n, hence superselection's discreteness, needs it.)*

65.4 Lecture C.4: the shared quotient, drilled

Build (24) by hand for two planes: write the four-dimensional real tensor product, the two-dimensional identification subspace, and the induced J on the quotient; verify $J^2 = -\mathbf{1}$ *on the quotient* and its failure on the full real product. Then redo Worked example III (the Bell state) tracking every identification, and compute the reduced covariance of one plane — mixed, as entanglement requires.

Problems.

1. For *unshared* structures $J_1 \neq J_2$, exhibit two representatives of the "same" product state whose difference is not in the identification span: the quotient is ill-defined, which is the Renou pathology structurally.
2. Show the swap operator descends to the quotient iff the structures are shared. *(Indistinguishability needs one i.)*
3. Extend (24) to three factors and verify associativity of the construction.

65.5 Lecture C.5: textures and the pentagonal lattice

A coupling texture as a U(1)-connection on the flavor graph: edges $|y_{ij}|R(\varphi_{ij})$, vertices with dials, re-zeroing as gauge. Fix a spanning tree of the 3×3 bipartite graph and null its edges; count the surviving loop angles ($(N-1)^2 = 4$); verify on a random texture that the four plaquette angles are re-zeroing invariant; then impose unitarity and watch the four collapse to one — the Jarlskog uniqueness, by hand (Worked example V supplies the block arithmetic). Close with the parent program's sharpening: with one circle and the integer $L = 5$, first-order angles live on $(\pi/5)\mathbb{Z}$, and the physical quark phase is the two-step lattice point $2\pi/5$ (Arisaka, 2026, A9-SM).

Problems.

1. Count loop angles for $N = 2$ and conclude CP conservation: the Kobayashi–Maskawa counting, graph-theoretically.

2. Show that a texture whose loop angles all lie in $\{0, \pi\}$ can be re-zeroed to real entries: achirality = realizability.

3. On the pentagonal lattice, list every achiral angle and verify that $2\pi/5$ is the *smallest* chiral one: nature's CP violation is minimal on its own alphabet.

65.6 Lecture C.6: the doors — winding arithmetic

Lemma 12.1 as a calculus. Tabulate, for a fermion of weight n: allowed bilinears ($\psi\bar{\psi}$ always; $\psi\psi$ iff $n = 0$), allowed trilinears with fields of weights m_1, m_2 (iff $n + m_1 + m_2 = 0$), and the induced conservation laws. Verify the Standard-Model instances: every Yukawa is fiber-neutral; the Majorana door opens only at the singlet; LHN is the unique portal touching it linearly. Conclude with the arithmetic slogan of Worked example VI: on the circle, conservation laws are integers.

Problems.

1. Classify all fiber-neutral *quartic* couplings among fields of weights $\{0, \pm\frac{1}{2}, \pm 1\}$ and match them to the Standard-Model quartics.

2. A weight-$\frac{1}{2}$ field is proposed. Show single-valuedness on the circle forbids it outright, or forces a double cover — and connect the alternative to the spinor double cover of Lecture B.1.

3. Prove: any global U(1) of a fiber-neutral Lagrangian that acts by dial shifts is automatically gauged by the circle. *(In this framework there are no ungauged dial symmetries to break.)*

66 An annotated reading guide

The bibliography, re-shelved as five reading paths, each in recommended order with a one-line brief. The guide assumes the graduate reader of the Preface.

Shelf 1 — the analysis (Course A's companions). Paley and Wiener (1934): the source of Theorem 13.3; read §§1–8 for the half-plane theory. Streater and Wightman (1964): chapters 2–3 for tube analyticity — the field-theoretic form of everything in Course A; the PCT chapter connects to T-P4-5.

Shelf 2 — the twistor school (Course B's companions). Penrose (1967): the founding paper; read for the geometry, not the formalism. Penrose and MacCallum (1973): the readable survey — the positive-frequency discussion in its §2 is the seam this paper builds on. Eastwood *et al.* (1981): the rigorous transform; theorem statements first, proofs on the second pass. Penrose and Rindler (1986): the reference volume for every spinor convention used in Course B. Penrose (1976): the nonlinear graviton — the open frontier, and objection O-P4-1's context. Witten (2004): the amplitudes renaissance; skim for how twistor variables re-entered mainstream particle theory.

Shelf 3 — the selection theorems. Ashtekar and Magnon (1975) and Kay (1978): the uniqueness of the positive-energy complex structure, imported at T-P4-2; Kay's paper is the sharper statement, Ashtekar–Magnon the geometric one. Read after Lecture C.2, whose argument they complete rigorously.

Shelf 4 — the infrared and the interacting frontier. Haag (1992): chapter II.4 for Haag–Ruelle theory (T-P4-8's engine); chapter VI for superselection — the three-superselections pattern of §39 in its original habitat. Buchholz (1977) and Faddeev and Kulish (1970): the two original footholds of §38, in that order. Fröhlich (1973): where the infraparticle began — the no-ground-state theorem the door theorem rides. Pizzo (2003) and Bachmann et al. (2012): the dressing constructions, iterated and cutoff-free — T-P4-9's middle rung. Chen et al. (2010): scattering with the clouds on, the strongest result under the door. Buchholz (1986): the flux observables that make the sector label physical. Wick et al. (1952): where superselection itself began — Proposition 39.1's first instance, in its original algebraic dress.

Shelf 5 — the experiments and the sources. Aharonov and Bohm (1959): the phase made physical. Renou *et al.* (2021): the shared-i mandate; read the supplementary material for the network construction. Weyl (1929) and Wu and Yang (1975): the gauge phase's birth and its bundle reading. Green and Schwarz (1984): the string road's anchor, for the three-roads comparison. Penrose (2004): the question itself, chapters 4–5 and 33 — best read twice, once before this paper and once after. The GG-Theory cohort papers are ordered for this purpose in §8.1; the shortest useful spine is Arisaka, 2026, A2-GG6 (the arena), then Arisaka, 2026, A8-QM (the real-bulk series), then Arisaka, 2026, A9-SM (the flavor holonomies).

Close of Part IX, in plain terms. The workshop closes: two courses walked the analysis and the geometry from scratch, a third drilled the paper's own theorems, the dictionaries fixed the symbols, and the reading guide arranged the literature into five ordered shelves. A student who has worked through to here holds every tool the paper uses — which is exactly the right condition in which to meet the next Part: the objections, forty-two of them, from seven communities, at full strength.

Part X
Common Objections and Responses

Forty-one adversarial readings — from the twistor school, quantum foundations, the mathematician, the particle theorist, the relativist and string theorist, the historian and philosopher, and the student — each answered with pointers into the paper

A paper that claims to relocate the complex numbers of physics invites — and should invite — attack from every community it touches. This Part collects the forty-two sharpest adversarial readings we could construct or have received, grouped by the reader most likely to raise each, and answers each with explicit pointers to the theorem, worked example, or named open item that carries the load. Where the honest answer is "not yet" or "not ours to claim," it is said in those words. The five objections of the earlier editions are retained verbatim within their categories.

67 The twistor school

Objection 1 (O-P4-1) (the twistor referee, on the googly)
"Holomorphy on $\mathbb{PT}$ is constitutive, not derived: the nonlinear graviton and the Ward correspondence are complex-geometric through and through, and your real story covers only the free correspondence."

Response. Correct on scope, and stated in the text: the bridge as proved covers the free massless correspondence, where the $\mathbb{PT}^+/\mathbb{PT}^-$ division is demonstrably the frequency splitting. The nonlinear constructions are named as open. But note what the free-level result already forces: if the division of twistor space is the causal-positive splitting for every free field, then the complex structure the nonlinear constructions deform is itself sourced in real causal data, and the burden of "constitutive" shifts. The program's conjecture is that the nonlinear twistor geometry will prove to be the integrable packaging of the same two sources; its proof or refutation is a well-posed piece of mathematics, not a matter of taste.

Objection 2 (O-P4-2) (the twistor geometer, on novelty)
"The correspondence between $\mathbb{PT}^{\pm}$ and frequency sign is in Penrose–MacCallum, page one. You have discovered our abstract."

Response. The correspondence is theirs, cited as such at every use (Penrose and MacCallum, 1973). What is claimed as new is not the correspondence but its *source*: that the division is generated by a real operator (the polar phase of the constitutional clock, T-P4-2/T-P4-6) whose own credentials — uniqueness from causality and positivity on a derived static arena — come from axioms that never mention $\mathbb{C}$. The twistor literature treats the division as a basic structure of a primitively complex space; this paper derives the splitting first, real-ly, and then identifies the twistor division with it (T-P4-7). Same seam, opposite direction of sewing — and the direction is the paper.

Objection 3 (O-P4-3) (the twistor geometer, on metaphysics)
" 'Direction of explanation' is not a mathematical statement. The equivalences are symmetric; your arrow is taste."

Response. The equivalences are symmetric; the *axiomatic dependencies* are not, and that asymmetry is checkable. The real side (circle, clock, positivity) is derived in the parent program from locality, causality, and an equivalence principle, with no complex input at any stage (Part III); the holomorphic side has, in its own literature, no derivation from anything — it is the starting point. An arrow between A and B is earned when A has independent support and B follows; that is a logical relation, not an aesthetic one. What would reverse it: a derivation of the circle and the clock *from* holomorphy, which we would read with genuine interest.

Objection 4 (O-P4-4) (the conformal skeptic)
"Nature is not conformally invariant — masses, Λ, running couplings. Your twistor level is a statement about an idealized world."

Response. Agreed, and the architecture absorbs it: the bridge is a ladder, not a single rung (Figure 4). The Hardy and tube levels (T-P4-6, §32) hold for massive, non-conformal matter — they need only the clock — while the twistor level is exactly the statement of what *more* is bought when conformal symmetry does hold, i.e. for the free massless sector. Where the twistor ideal experiences mass as breaking, this framework locates mass structurally (the scale fiber; T-P4-8's gap hypothesis), so the ladder's rungs are regimes, not approximations. The idealized world is a sector of this one, and the theorem says which.

Objection 5 (O-P4-5) (the twistor program historian)
"Penrose has emphasized the positive-frequency property of $\mathbb{PT}^+$ for fifty years as evidence that holomorphy is fundamental. You cite his observation as your theorem."

Response. His observation, his emphasis, and his evidence — all credited. The step he did not take, because his program reads the arrow the other way, is the one this paper takes: exhibiting a *real, axiomatically derived generator* for the splitting and proving its uniqueness (T-P4-2, via Ashtekar and Magnon, 1975; Kay, 1978), so that the positive-frequency property becomes evidence that *causal positivity* is fundamental and holomorphy its large-symmetry face. The facts are shared; the theorem T-P4-7(iii), identifying his division with our generator's splitting level by level, is the new object on the table.

Objection 6 (O-P4-6) (the ambitwistor/amplitudes practitioner)
"Modern twistor methods live in momentum space and amplitudes, not in your position-space Hardy theory. Your bridge connects to 1977, not to 2026."

Response. Partly fair. The bridge as proved addresses the classical correspondence, which is also where Penrose's *question* lives; the amplitudes renaissance (Witten, 2004) is engaged in the three-roads comparison as the twistor column's modern strength on interactions. But the connection is not antiquarian: momentum-twistor variables inherit exactly the positive-energy structures whose real sourcing is this paper's content, and T-P4-8's asymptotic route is the natural place where an amplitudes-level statement would attach. We record that extension as open and worth a specialist's attention — ideally a skeptical one.

Objection 7 (O-P4-7) (the twistor geometer, on the circle)
"Twistor space has no sixth dimension and no phase circle. If your circle were fundamental, $\mathbb{PT}$ would know about it."

Response. $\mathbb{PT}$ does not need to know about the circle, because the two sources have divided labor (T-P4-3): the circle owns the *gauge* half (charge, quantization, superselection), the clock owns the *state-space* half — and twistor geometry is the conformal packaging of the clock's half only. That is why twistor methods are at their best for uncharged structure (massless free fields, self-dual geometry) and need extra apparatus for charge. On the bridge's account this is not an accident of the formalism but a portrait of the source-split itself; the circle's fingerprints appear exactly where twistor constructions import gauge bundles by hand.

68 Quantum foundations

Objection 8 (O-P4-8) (the real-quantum referee)
"Real quantum mechanics is experimentally dead. A 'real-bulk' account is refuted before it starts."

Response. What is dead is real quantum mechanics with *unshared* complex structures — real tensor products of separately-real spaces (Renou *et al.*, 2021). What this preprint derives is standard complex quantum mechanics, with one globally shared J built from one shared circle and one shared causal order; the complex tensor product is recovered as the real one quotiented by the shared-J identification. The experiment demands exactly the sharing this account supplies, and is cited here as its strongest external support, not as a threat survived.

Objection 9 (O-P4-9) (the operator theorist, after Stone)
"Stone's theorem already ties i to time: every strongly continuous unitary group is e^{itH}. You have dressed a 1932 theorem in six dimensions."

Response. Stone's theorem (Stone, 1932) *presupposes* the complex Hilbert space and then relates its given i to generators; it answers "how do i, t, and H interlock?" inside complex quantum mechanics. This paper answers the prior question: why is the state space complex at all, and what selects its J? The construction runs on *real* data (real solution space, real symplectic form, real generator), produces J by polar decomposition, and proves it unique from positivity (T-P4-2) — and only then does Stone's theorem become available, with its i now a theorem rather than an axiom. The 1932 result is downstream scaffolding, gratefully used.

Objection 10 (O-P4-10) (the reconstruction theorist)
"Operational reconstructions (Hardy; Chiribella–D'Ariano–Perinotti; Solèr) already single out complex QM from information axioms. Your geometry is redundant."

Response. The reconstructions (Chiribella *et al.*, 2011; Hardy, 2001; Jordan *et al.*, 1934; Solèr,

1995) are convergent evidence, not competition — but note what kind of answer they give: complex QM is selected *among state-space theories* by axioms about information processing. They locate $\mathbb{C}$'s necessity; they do not locate $\mathbb{C}$ itself in the physical world. This paper's claim is of the second kind: here is the circle, here is the clock, here is the operator (5) built from them, and here is the experimental fingerprint of their being globally shared (Renou *et al.*, 2021). A student can hold the reconstruction theorems in one hand and Part IV in the other; the first says "it had to be $\mathbb{C}$," the second says "and this is what $\mathbb{C}$ was."

Objection 11 (O-P4-11) (the curved-space field theorist)
"One-particle structures, taming, Ashtekar–Magnon, Kay: this is Wald's textbook, chapter 4. What exactly is yours?"

Response. The apparatus is the standard one (Wald, 1994) and cited line by line; three things are ours. The *arena*: standard theory selects J *given* a stationary spacetime; here staticity is a theorem of the axioms (Theorem 0), so the selection acquires constitutional standing rather than background-dependence. The *pairing*: the coincidence of the clock's J with the circle's on charged sectors, with antiparticles as the consistency condition (T-P4-3), has no textbook counterpart. And the *assembly*: carrying the same derived J through discrete symmetries, flavor holonomy, tensor structure, and the twistor division as one object is the paper. Imported bricks, new building — with the bricks' provenance stamped on each.

Objection 12 (O-P4-12) (the algebraic field theorist)
"Your three-superselections pattern is Haag's book, chapter VI. Superselection theory needs no circles."

Response. The theory of superselection sectors is indeed the algebraists' (Haag, 1992, gratefully shelved in the reading guide); what this paper adds is a *uniform geometric mechanism* for the three instances it meets: in each case (charge, horizon vacua, infrared clouds) the unobservable is a relative phase between sectors, and in each case the obstruction is an unreadability — of the fiber angle, of the port assignment, of the cloud at infinity — while the derived J passes through untouched (§39). The abstract theory says sectors happen; the geometric account says *where*, and why the i never falls into the cracks between them.

Objection 13 (O-P4-13) (the foundations referee, on measurement)
"Your 'real records' rest on the parent program's attractor story of measurement — a hidden-variable theory with an unproven typicality conjecture. Your arc is only as real as that."

Response. The dependence is disclosed in §21, conjecture named. But mark what this paper actually needs from the measurement layer: only that registered outputs are real quantities — counts, angles, rates — which is an operational fact independent of any interpretation, Copenhagen's included. The GRIP account supplies the parent program's mechanism for it; the arc of §22 survives with any mechanism, because no interpretation on the market registers a complex number. The typicality conjecture bounds the parent program's claims; it does not touch the theorem that the middleware's i is derived.

Objection 14 (O-P4-14) (the information theorist)
"Real, quaternionic, and complex QM differ operationally in known ways beyond network Bell tests. Why is one experiment your anchor?"

Response. Because it is the cleanest *exclusion*: Renou *et al.* (2021) turned the real-tensor-product alternative from a foundational preference into a falsified theory, which is the strongest statement experiment ever makes. The broader landscape (single-system real simulations exist; quaternionic composites over-constrain) is consistent with, and summarized by, the same structural point the quotient theorem makes (T-P4-4): what nature demands is not "complex numbers" but *one shared* complex structure, which is exactly what two global real sources supply and what unshared real or over-rigid quaternionic structures cannot. The one experiment is the anchor because it tests precisely the sharing.

69 The mathematician

Objection 15 (O-P4-15) (the analyst, on rigor)
"These are physicist's proofs: dense domains, self-adjointness, and convergence are waved through."

Response. Guilty as charged, deliberately and with a map. The register is stated in §44: kinematic-level proofs, with the functional-analytic care delegated — explicitly, at each use — to the classical sources where it is done once and well (Paley and Wiener, 1934 for the Hardy theory; Kay, 1978 for the selection theorem; Haag, 1992 for scattering). Every claim that is ours reduces to finite-dimensional linear algebra (weights, quarter-turns, quotients) or to an identification between two rigorously established structures. A mathematician who wants the ϵ's will find each one in the cited page ranges of the reading guide; none of the paper's content lives in the gap.

Objection 16 (O-P4-16) (the analyst, on the transform)
"$\mathsf{H}^2 = -1$ fails on constants; your J is not a complex structure on all of L^2-adjacent spaces you use."

Response. Correct, tracked, and physical: the zero-frequency kernel is excluded wherever J_A is asserted (the gapped hypothesis of Lemma 11.1; the measure-zero statement for massless fields in infinite volume; O4 in the ledger), and Lecture A.6 quantifies the failure modes rather than hiding them. The exclusion is not a technical embarrassment but a physical classification: the modes on which the clock defines no orientation are exactly the non-dynamical ones (constants, finite-volume zero modes), which is why they surface in physics as superselected or classical. The domain bookkeeping and the ontology agree.

Objection 17 (O-P4-17) (the algebraist, on the quotient)
"Equation (24) is stated for algebraic tensor products; completions, continuity of the induced J, and infinite dimensions need proof."

Response. For the finite-mode statements this paper makes (Worked example III; Lecture C.4) the algebra suffices and is complete as given. The infinite-dimensional upgrade is standard once

J is orthogonal/symplectic and bounded — which the polar phase is, by construction, away from the excluded kernel — so the quotient extends to the Hilbert completion with the induced J isometric. We assert nothing sharper than that, and the second-quantized statements ride on the standard Fock functor. A careful write-up belongs in a mathematical-physics venue's appendix; nothing conceptual awaits it.

Objection 18 (O-P4-18) (the geometer, on cohomology)
" 'Čech in one page' with two patches is not sheaf cohomology; the transform's naturality, and everything beyond H^1 of a line, is elided."

Response. It is exactly as much cohomology as the paper uses, and the boundary is marked: Lecture B.6 computes on the two-patch cover of a projective line, which suffices for the elementary states and for the conceptual point (only overlap data are physical), while the rigorous, natural, any-cover formulation is cited where it lives (Eastwood *et al.*, 1981). We would resist the suggestion that the one-page version misleads: it is the computation every twistor course actually does first, and its moral — cocycles modulo coboundaries as the holonomy logic of T-P4-5 transplanted — is the honest content of the machinery at this paper's resolution.

Objection 19 (O-P4-19) (the operator theorist, on polar decomposition)
"$A(-A^2)^{-1/2}$ requires A skew-adjoint with dense range on a specified space; 'generator of the dynamics' is not a definition."

Response. For every use in this paper A is either a finite real matrix (the worked examples; the lattice checks) or the standard Klein–Gordon/Dirac generator on Cauchy data, whose skew-adjointness and spectral calculus are textbook; the gapped hypothesis gives bounded invertibility of $|A|$, and Lecture A.6/O4 fence the rest. The phrase in Lemma 11.1 is shorthand for exactly this class, stated in its preamble. The referee's demand is met by restriction, not by evasion: we claim the formula on the class where it is classical, and the physics never asks for more.

Objection 20 (O-P4-20) (the historian of mathematics)
"Attributions are romantic: Paley–Wiener theorems predate and postdate 1934 in many forms; 'Euler wound them around a circle' is potted history."

Response. The survey register of Part I is signposted as such, the chronology table (Table 1) states its episodes at one-line resolution, and the load-bearing citations are to specific results, not eras. Where a name covers a family (Paley–Wiener), the theorem statement in the text (Theorem 13.3) is the precise member used. Corrections to the historical color would be received gratefully and would change no theorem; the mathematics is anchored to the cited statements, not to the anecdotes.

70 The particle theorist

Objection 21 (O-P4-21) (the regime referee)
"Your theorems are free-field theorems. The world is interacting, massless-infrared-

infested, and gravitating; your i is proved for a world that does not exist.”

Response. The extension ledger is in Part VI and is exactly as strong as the field’s own frontier: interacting gapped matter, conditionally closed by the asymptotic route; confined sectors, covered at the hadronic level; stationary horizons, discharged with thermality as the partial trace; the interacting massless infrared, closed at the rigorous-model level by the door theorem (T-P4-9): the sector label is instrument data, the soft cloud never touches J, and what remains open is the full-QED existence theory, inherited by name. No claim in this preprint outruns that ledger, and the ledger outruns, at present, any competing account of where the complex structure comes from — including the account on which it comes from nowhere.

Objection 22 (O-P4-22) (the infrared referee)
“Your door theorem is a theorem about models. The electron of the real world is an infraparticle of full QED: no sharp mass shell, velocity superselection, and a dressing construction (Faddeev–Kulish) that after half a century is still not rigorous. You have proved the easy rungs and declared the ladder climbed.”

Response. The ladder is declared, not climbed — that is why the theorem states it as a ladder. What T-P4-9 claims is scoped in its own statement: exact at the van Hove stratum; at Nelson/Pauli–Fierz, riding imported theorems (Fröhlich, 1973; Pizzo, 2003; Bachmann et al., 2012; Chen et al., 2010) whose hypotheses are displayed; at full QED, a reading carried with its status shown, inheriting the field’s own open existence problem by name. But notice what the objection is about: whether the dressed states *exist* at full rigor — an existence question about particles. It is not about where i comes from. The two clauses that answer this paper’s question — the sector label is a c-number port datum, and the coherent dressing is affine, hence J-blind — are statements about coherent dressings as such, and they hold for any dressing the full theory will ever construct, Faddeev–Kulish included. If the QED frontier moves, the door theorem’s regime moves with it; nothing in this paper is hostage to the schedule.

Objection 23 (O-P4-23) (the QCD skeptic)
“ ‘The hadronic spectrum is gapped’ does not make Haag–Ruelle applicable to QCD: confinement, color, and the absence of a rigorous construction all stand in the way.”

Response. Correct, and the claim is scoped accordingly: T-P4-8 is conditional on (H1)–(H3) *for the physical spectrum*, and its QCD reading is a consistency observation — the asymptotic objects instruments meet are gapped hadrons, so the hypotheses attach where the operational dictionary needs them — not a claim that four-dimensional QCD has been rigorously constructed (it has not, for anyone). The honest summary: in the regime where any interacting field theory can currently be said to have a particle interpretation, the derived J is its complex structure. The construction problem itself is the millennium-grade frontier, shared by all.

Objection 24 (O-P4-24) (the flavor phenomenologist)
“$2\pi/5 = 72^\circ$, but the fits give $\delta \approx 66$–69°. Your ‘lattice-locked’ phase is off by several degrees.”

Response. The pentagonal value is the parent program’s *first-order* statement, quoted here as such, with the quantitative-completion step and its comparison to data carried in Arisaka, 2026, A9-SM — where the residual is treated within the program's stated theory-uncertainty model, and where the falsifier is registered: a measured drift outside the lattice's completion window retires the account. This paper's own use of the lattice is structural (T-P4-5(iv): angles live on $(\pi/5)\mathbb{Z}$ at first order, so CP violation is not a continuous dial), which the few-degree completion question does not touch. Precision belongs to the parent paper; the discreteness claim is what is staked here.

Objection 25 (O-P4-25) (the flavor theorist, on novelty)

"Rephasing invariants and the Jarlskog construction are forty years old. 'Flavor holonomy' renames known lore."

Response. The invariants are classical; two things are not. First, the *realization*: rephasing as literal dial re-zeroing on a physical circle, so that Jarlskog invariance is holonomy invariance in the same sense as Aharonov–Bohm — a statement with content, since it makes CP violation a chirality of real texture and CP itself an orientation reversal (T-P4-5), predictions of structure rather than notation. Second, the *alphabet*: with one circle and $L = 5$, the loop angles are lattice-valued, which the classical lore cannot express because it has no geometry for the phases to be locked *to*. Worked example V lets a skeptic run the whole computation with no i and judge whether "renames" survives the experience.

Objection 26 (O-P4-26) (the collider pragmatist)

"No cross sections, no spectra, no discovery reach. What measurement does this paper change?"

Response. None — by design, and the design is stated (§44): this paper reorganizes the *foundations* of structures whose phenomenology lives elsewhere in the cohort (the lock cluster in Arisaka, 2026, P1-KK; the flavor numbers in Arisaka, 2026, A9-SM; the cosmology in Arisaka, 2026, A10-Cosmos). Its falsifiable exposure is listed in §48 and is real but indirect: shared-i tests, lattice drift, neutrino nature. A reader who counts only cross sections should read the cohort papers; a reader who has ever wondered why the Lagrangian is complex will find here the only answer on offer that could, in principle, have been false.

Objection 27 (O-P4-27) (the effective-field theorist)

"Below any cutoff, I write the most general complex Lagrangian and match. Your real structures are invisible in the EFT, hence physically empty."

Response. Two of the real structures are visible *inside* the EFT precisely as its unexplained rigidities: the integrality of charge (why the EFT's U(1) representations are quantized at all) and the discreteness of CP phases on the pentagonal completion (why one coupling is not a free dial). A third is visible at the EFT's edge: the shared-i constraint that network experiments test. EFT is the right tool for dynamics below a cutoff and the wrong tool for the question this paper asks, which is why the question survived eighty years of EFT competence untouched: matching coefficients cannot tell you what the number system is *made of*.

71 The relativist and the string theorist

Objection 28 (O-P4-28) (the string theorist, on fairness)
"Your three-roads table understates us: worldsheet holomorphy is not 'primitive' — it emerges from Polyakov quantization; and strings derive far more of physics than either other road."

Response. The table's entry is scoped to this paper's single question — where $\mathbb{C}$ enters — and on that question the worldsheet's complex structure arrives with the Riemann-surface formulation of the path integral, however motivated: it is not derived from a causal-real substrate within the program, which is all "primitive" means in that column. On breadth we defer without irony: the comparison paragraph and Green and Schwarz (1984) record the string road's structural rigidity on its own terms, and the parent program's fuller comparison (Arisaka, 2026, A2-GG6, Remark 0.4) frames the two economies as premises placed at opposite ends of the energy axis, with the reader left to weigh them. No adjudication is made here beyond the one question asked.

Objection 29 (O-P4-29) (the relativist, on non-spatial dimensions)
"A 'dimension' with no metric role in propagation is a fancy name for an internal space. Kaluza–Klein plus fiber bundles already said everything your circle says."

Response. The circle *is* a principal fiber — that is T-P4-1's language — and the bundle formulation (Wu and Yang, 1975) is its acknowledged precedent. The added content is twofold. Constitutionally: the fiber is *forced* (Theorem 0's two-sided count), not chosen, and carries the ledger's arithmetic (§18). Physically: unlike Kaluza–Klein's spatial circle, it is unreadable in principle (no light cone along it), which converts "internal" from a modeling word into a theorem-grade property with consequences — superselection, the Born form's marginalization, the absence of angular satellites at the lock. An internal space that must exist and cannot be looked at is not merely renamed; it is located.

Objection 30 (O-P4-30) (the relativist, on rotation and collapse)
"Kerr has no global timelike Killing field and collapse is not stationary. Your clock-selection theorem dies exactly where gravity gets interesting."

Response. Named, fenced, and partially discharged rather than hidden: the stationary-exterior case is settled with the vacuum multiplicity as port assignments and thermality as the partial trace (§39; Worked example IV); the Kerr anchor and the collapse-epoch factorization are closed at the model level in the parent treatment (Arisaka, 2026, A8-QM, entries R14–R15): the rotating-frame selection fails exactly where the Kay–Wald obstruction (Kay and Wald, 1991) says it must — physics, not bookkeeping — so the anchor moves to the asymptotic frame and superradiance becomes a symplectic pumping channel, while the collapse epoch factorizes into a universal thermal channel times a decaying transient; what stays open, and is listed in §49, is the exact curved-space existence theory. Meanwhile the half of J gravity cannot touch is a theorem: stage curvature preserves the fiber block, so the circle's structure — and everything gauge about $\mathbb{C}$ — is immune. The interesting region is open on the clock's side only, and exactly as open for every neighboring program.

Objection 31 (O-P4-31) (the quantum-gravity theorist)
"No graviton, no quantized geometry, no Planck-scale statement: a paper about the deep structure of physics that avoids quantum gravity."

Response. Avoidance is the parent program's architecture, inherited knowingly: geometry there is not quantized — the equivalence principle is generalized instead (AX-2), quantization enters at the boundary dictionary (AX-3), and the cohort's gravity-sector statements live in their own papers. This paper's question is well-posed and answerable *within* that architecture, and its answer would survive translation: any future account in which causal order, positivity, and a compact gauge fiber persist at the fundamental level will inherit the derivation of J wholesale. If quantum gravity abolishes those three, it abolishes far more than this paper.

Objection 32 (O-P4-32) (the cosmologist)
"Your 'globally static arena' is falsified by the sky: the universe expands."

Response. The static metric of Theorem 0 is the constitutional *bulk* background; observed expansion is carried, in the parent program, by the boundary/effective description whose cosmology is Arisaka, 2026, A10-Cosmos's subject — and the interface between the canonical bulk J and observer-adapted cosmological dictionaries is exactly the port structure of §20, with cosmological particle creation as its channel form. What this paper needs from staticity is the uniqueness clause of T-P4-2 at the constitutional level, not a claim that comoving observers see a static sky; the distinction is the same bulk/boundary split the whole cohort runs on.

72 The historian and the philosopher

Objection 33 (O-P4-33) (the historian referee)
"Paley–Wiener, tube analyticity, and the Penrose transform are all classical. You have written a review and called it a derivation."

Response. The components are classical and cited as such; the claims that are new are identifications, and identifications are where explanations live. That the Hilbert transform — selected by causality and positivity as the unique invariant complex structure on a static arena — is the *same* operator that generates the Hardy splitting; that the twistor division restricts on every timelike line to exactly that splitting; that the antiparticle pairing is the consistency condition between this J and the gauge circle's: none of these is in the classical literature as a statement about where the complex structure of physics comes from. A review arranges known results; this argument uses them to reverse a direction of explanation that has stood since 1926.

Objection 34 (O-P4-34) (the Platonist referee)
"You have moved the magic, not dissolved it. The circle's harmonic analysis is beautiful complex mathematics; the Platonic mystery stands."

Response. Agreed, and conceded in §47 without reservation. The preprint's claim is about nature's inventory, not mathematics' worth: no complex primitive is instantiated; the structures instantiated are a circle, a clock, and orientations; the complex numbers are the observer's compression of these. Why that compression is itself a mathematical object of extraordinary depth is a question about the Platonic world, and the author declines to pretend that a physics theorem answers it. Half the magic was physics; that half is hereby geometry. The other half was never ours to dissolve.

Objection 35 (O-P4-35) (the philosopher, after Hamilton)
"Hamilton reduced $\mathbb{C}$ to ordered pairs in 1835. Your 'real presentation' adds physics-flavored decoration to a done deed."

Response. Hamilton showed the reduction is *possible*; the chronology section says so and salutes him. What he could not supply — what no one could, before there was a physics of the pairing rule — is the *selection*: which quarter-turn, why shared, why unique. Those are exactly the theorems here (T-P4-1's commutant, T-P4-2's positivity, T-P4-4's sharing), and they are where the physical content lives: an unselected real presentation is notation, a selected one is a discovery about the world. The paper's one-line relation to 1835 is stated in Part I: this is the physics that was missing from Hamilton's algebra.

Objection 36 (O-P4-36) (the philosopher of science, on underdetermination)
"Formulations are cheap: real, complex, quaternionic presentations of QM all exist. Choosing the real one and calling it ontology is underdetermined."

Response. Mere reformulation would be; this is not that, on two counts. Empirically, the presentations are *not* equivalent at the composite level: the unshared-real alternative is falsified (Renou *et al.*, 2021), and the quaternionic route over-constrains composites — the data thin the herd. Structurally, the claim is not "the real presentation is prettier" but "the complex structure is a *derived object* with real sources and a uniqueness theorem," which is a statement about explanatory architecture, testable by its consequences (antiparticle pairing, discrete-symmetry table, lattice phases) rather than by taste. Underdetermination applies to distinctions without consequences; §48 lists the consequences.

Objection 37 (O-P4-37) (the philosopher of mind, on the compression thesis)
" 'Humans invented $\mathbb{C}$ because we are 4D-blind' is unfalsifiable psychology dressed as physics."

Response. The thesis is offered as history's explanation, not as a theorem, and it is flagged as such (§47); the theorems are the compression *facts* it narrates: fiber-blind observers must package weight pairs (T-P4-1), tube-embedded observers must package frequency pairs (T-P4-6), and both packagings are unique. The psychological gloss is as falsifiable as counterfactual history ever is — which is to say, it is an interpretation — but it is disciplined by the mathematics: had the sources been visible, the compressed notation would have been optional, as phasors are. We commend the objection's standard and accept its cost: one paragraph of the paper is narrative, and says so.

Objection 38 (O-P4-38) (the Wigner scholar)
"Dissolving 'unreasonable effectiveness' via one case study is cheap; Wigner's puzzle was about the whole pattern of mathematics' success."

Response. Agreed, and the conclusion claims only the case: *one* of the pattern's sharpest instances — the algebraists' invention running the quantum world — converts, on this account, from "unreasonable" to "unrecognized" (§58). Whether the whole pattern so converts is the parent program's more ambitious wager, engaged in Arisaka, 2026, A11-Origin alongside Wigner, Tegmark, and Penrose's three worlds, and deliberately not spent here. A pattern is dissolved one load-bearing instance at a time; we submit this as the instance, and leave the pattern to the paper whose job it is.

73 The student

Objection 39 (O-P4-39) (the graduate student, practically)
"If complex numbers are 'only' compression, why does every computation use them? Should I stop?"

Response. Certainly not — compression is what good notation *is*. You should keep using $\mathbb{C}$ exactly as you keep using center-of-mass coordinates: because it is the optimal encoding of two real structures you would otherwise carry by hand. What the paper changes is not your calculations but your answers to "what am I doing when I write $e^{i\theta}$?" — and, occasionally, your reach: the real presentation is the right frame precisely where the standard one mystifies (why antiparticles, why antiunitary T, why superselection, why $\mathbb{PT}^{\pm}$), as the worked examples are designed to show. Use the shorthand; own the longhand.

Objection 40 (O-P4-40) (the student, on predictions)
"Does any experimental number come out different? If not, isn't this all interpretation?"

Response. Within established quantum mechanics, no number changes — deliberately: a derivation of the formalism that altered its predictions would be a refutation, not an explanation. The paper's exposure is at the edges (§48): shared-i network tests, the pentagonal completion window of the CP phases, the Majorana question — plus one retroactive prediction of the strongest kind: that the real-Hilbert-space alternative *had* to fail, for the stated geometric reason, before the experiment was run. "Interpretation" is the wrong shelf: this is a derivation with named ways to be wrong, which is the definition of physics done to foundations.

Objection 41 (O-P4-41) (the student, on authority)
"My textbooks say the complex Hilbert space is a postulate. You are one preprint. Why believe you?"

Response. Do not believe; check — the paper is built for it. Every theorem reduces to computations you can do (the courses and problem sets exist for exactly this), every import is cited to a page range, and the verification suite re-derives each symbolic claim mechanically. Your textbook is right that the standard formulation postulates $\mathcal{H}$ (von Neumann, 1932); the question this paper adds is whether the postulate has a source, and its answer is a chain you can audit link by link in a week. Authority is the wrong tool on both sides; Lecture C.2 and a rainy afternoon are the right ones.

Objection 42 (O-P4-42) (the student, on Penrose)
"What do you expect Penrose himself would say?"

Response. We would not presume to script him; the paper is arranged for his scrutiny rather than his applause. Our expectation, stated plainly: the two-sources split, the antiparticle handshake, and the identification of $\mathbb{PT}^{\pm}$ with a derived splitting would engage him, because two of the three are built from his own theorems; the direction of explanation he would contest, as objection 3 records; the googly row he would note first, as objection 1 concedes; and the Platonic remainder he would keep, with our agreement, as objection 33 grants. If the paper earns from him one sentence of the form "the identification is correct; the reading is not mine," it will have done its work: the mathematics shared, the question finally joined.

Closing note. These forty-two objections are the strongest we could assemble against a paper whose thesis invites them from seven directions at once. The pattern of the answers is the paper in miniature: what is proved is pointed to; what is imported is credited; what is open is named (the googly, the full-QED infrared, the Kerr anchor, the typicality conjecture upstream); and what is conceded — the Platonic wonder — is conceded without regret. A reader who finds an objection missing from this list is invited to send it; the list has grown by exactly that mechanism. What remains is the paper's foundation in the literature: the references, in two groups, each with a note on what it established and how it is used here.

Part XI
References

Cited works, each with a relevance note

The references are organized in two groups: the GG-Theory cohort papers, among which this preprint stands in the Foundations Branch, and all other works. Every entry carries a relevance note naming the section where it is first used.

This paper cites its cohort siblings in completed tense: each result is imported as established in its home paper, with hypotheses restated here where they bear weight.

The second group is deliberately short: the classical mathematics on which the bridge stands, the twistor school's own theorems, and the experimental anchor.

A. Papers by Katsushi Arisaka (the GG-Theory cohort)

K. Arisaka. *The GG-Constitution: Four Axioms and Seven Postulates of the GG-Theory Program, From Which GG-6, BI-6, OGN-5, and GDOS Descend as Theorems*, Arisaka-GG-A1-Constitution. Preprint, available at https://preprints.arisaka-gg.org/a1-constitution/.
Relevance. The constitutional foundation of the entire program: four axioms and seven postulates from which everything downstream is derived. This paper draws on it constantly and specifically — AX-1 supplies the causal time orientation, AX-2 the real composite connection whose complex-free character Part III verifies clause by clause, AX-3 the operational dictionary whose positivity clause selects the sign of the derived J, and AX-4 the closure that forbids unforced structure. The axioms are rendered at full resolution in §16. First cited in §8.1.

K. Arisaka. *GG-Theory with GG-6: Geometry–Gauge Locking at the 6D Bulk (GG-6) at the Heart of Grand Geometric Theory (GG-Theory)*, Arisaka-GG-A2-GG6.
Preprint, available at https://preprints.arisaka-gg.org/a2-gg6/.
Relevance. The arena paper: Theorem 0 derives the six-dimensional bulk $\mathcal{M}_4 \times I_r \times S^1_\theta$ with its globally static metric from AX-1 and AX-2 alone, establishing both of this paper's real sources — the compact phase circle and the constitutional clock — as forced geometry rather than postulated scaffolding. Its six-step proof skeleton, its non-spatiality classification by generators, and its two-sided minimality are summarized in §17 and used by every theorem of Parts IV–V. First cited in §8.1.

K. Arisaka. *Anomaly Freedom of BI-6: The Modified Bianchi Identity (BI-6) for the Six-Force, Six-Dimensional Bulk (GG-6)*, Arisaka-GG-A3-BI6.
Preprint, available at https://preprints.arisaka-gg.org/a3-bi6/.

Relevance. The ledger paper: proves that the five integers $(M, N, L; J, K) = (1, 3, 5; 7, 35)$ are forced by the anomaly-closure architecture, each with a primary structural meaning. Two entries do direct work here: $M = 1$, the unique tensor seam, is the winding unit that makes charge quantization integral and superselection discrete (T-P4-1); and $L = 5$, the defining unified block, is the pentagonal alphabet on which the CP-violating loop holonomies of T-P4-5 are lattice-locked. Summarized in §18. First cited in §8.1.

K. Arisaka. *From Euclid to GG-6: The Realization of Euclid's Elements in Six Dimensions — GU-5, OGN-5, and a Parameter-Free Reconstruction of Modern Physics*, Arisaka-GG-A4-Euclid. Preprint, available at `https://preprints.arisaka-gg.org/a4-euclid/`.
Relevance. The lineage paper: traces the geometrization program from Euclid through Riemann, Minkowski, Einstein, Weyl, Kaluza–Klein, and holographic renormalization to the present framework, establishing the historical pattern — a compulsory structure recognized as geometry and absorbed — of which this paper's move, returning the complex numbers themselves to geometry, is offered as the latest instance. The historical Part of this paper is a complex-numbers-specific companion to its broader narrative. First cited in §8.1.

K. Arisaka. *GDOS: Geometric Dynamics of Open Systems — 6D GG-6 Bulk Projected to 4D Boundary for Quantum Measurement, General Relativity, and QFT*, Arisaka-GG-A5-GDOS. Preprint, available at `https://preprints.arisaka-gg.org/a5-gdos/`.
Relevance. The boundary-law paper: constructs the operational dictionary demanded by AX-3 and derives the open-system runtime — the forced projection, its always-on character, the exact memory-kernel reduction, complete positivity, the GKLS generator, and the seven textbook quantum postulates as theorems. This paper uses its architecture at every boundary: the port structure that houses the vacuum multiplicity and the infrared sectors (§38, §39), and the two-fibers-two-halves split summarized in §20. First cited in §8.1.

K. Arisaka. *The GRIP on GDOS — GRIP: the Geometric Engine of Symmetry Breaking, from the Big Bang to Mind*, Arisaka-GG-A6-GRIP.
Preprint, available at `https://preprints.arisaka-gg.org/a6-grip/`.
Relevance. The runtime paper: builds the four-pillar system — the total geometric–gauge object, the covariant conservation identity with its four Ward currents, the geometric free-energy functional, and GRIP, the cascade by which boundary configurations descend to the ground attractor. Its relevance here is twofold: the coupling textures whose loop holonomies measure CP chirality (T-P4-5) are GRIP-stabilized outputs rather than free functions; and the actualization story it supplies is why every registered record in this paper's arc is a real quantity (§21). First cited in §8.1.

K. Arisaka. *The Qi-4 Pillars — Qi-QUA, Qi-EP, Qi-CON, Qi-HAL: The Quantum-Information Backbone of GDOS*, Arisaka-GG-A7-Qi4.
Preprint, available at `https://preprints.arisaka-gg.org/a7-qi4/`.
Relevance. The information backbone of the boundary law: the four informational pillars that complement the geometric ones of the runtime paper, fixing how boundary information is counted,

conserved, and registered. It enters this paper implicitly wherever the operational dictionary's instruments are treated as information-bearing boundary data — the reading that lets vacuum choices, port assignments, and soft-cloud sectors be classified as dictionary freedom rather than bulk ambiguity. First cited in §8.1.

Arisaka, K. (2026). *Quantum Mechanics Derived from GG-Theory as Projection from 6D Bulk to 4D Boundary.* Arisaka-GG-A8-QM preprint.
Preprint, available at https://preprints.arisaka-gg.org/a8-qm/.
Relevance. The parent treatment and this paper's closest companion: the quantum-mechanics paper whose real-bulk series RB-1–RB-12 states, at discussion level, the results this paper expands, re-proves, and re-derives for a broader readership — the two sources, the antiparticle handshake, the discrete-symmetry table, the interacting completion, and the holomorphy bridge. Its three-address analysis of "the Hilbert space," its measurement resolution, and its Penrose engagement stand behind Parts IV–VII throughout. First cited in §8.1.

K. Arisaka. *The Standard Model as Theorem of GG-Theory: Its 28 Parameters as Observations of One Six-Dimensional Geometry*, Arisaka-GG-A9-SM.
Preprint, available at https://preprints.arisaka-gg.org/a9-sm/.
Relevance. The Standard-Model paper: derives the 28-parameter data set, and specifically — for this paper's purposes — the flavor results quoted in T-P4-5 and Worked example V: the CKM block as a projector-misalignment cascade, the pentagonal interference structure with the first-order Dirac phase $\delta = 2\pi/5$, the discrete-symmetry theorems T-SM-46.1/46.2 (this paper's R8/R9 in their theorem form), and the dial-free-door mirror T-SM-44.1. Its theory-uncertainty model carries the precision comparison that objection O-P4-24 defers to. First cited in §8.1.

K. Arisaka. *The Standard Cosmology as Theorem of GG-Theory: The Cosmic Inventory as Observations of One Six-Dimensional Geometry*, Arisaka-GG-A10-Cosmos.
Preprint, available at https://preprints.arisaka-gg.org/a10-cosmos/.
Relevance. The cosmology paper: carries the dial-free-door theorem in its cosmological form (T-Cosmos-18), the forced Majorana channel, TeV-scale resonant leptogenesis at the lock, and the observer-indexed-vacuum treatment whose port reading this paper generalizes in §39. Cited in the falsifiability section for the neutrino-nature exposure: an established Dirac nature for the light neutrinos would retire its leptogenesis module and, with it, one of the named ways this account can fail. First cited in §8.1.

K. Arisaka. *The Geometric Inevitability of Cosmos, Life, and Mind: The Self-Referential Closure of the GG-Theory Program*, Arisaka-GG-A11-Origin.
Preprint, available at https://preprints.arisaka-gg.org/a11-origin/. No result of the present paper depends on it.
Relevance. The philosophical summit of the cohort: the paper that engages Wigner's unreasonable effectiveness, Penrose's three worlds, and Tegmark's mathematical universe, and that carries the calibrated geometrization slogan this paper's conclusions echo. The division of labor is deliberate: this paper dissolves one load-bearing instance of the effectiveness puzzle (the complex numbers),

while the pattern-level engagement — whether all such instances so dissolve — is that paper's more ambitious wager, deliberately not spent here (objection O-P4-38). First cited in §8.1.

Arisaka, K. (2026). *The GG-6 Lock Cluster: A Non-Spatial Kaluza–Klein Resonance near 3.85 TeV as the Standing LHC Falsifier of GG-Theory*. Arisaka-GG-P1-KK preprint.
Preprint, available at https://preprints.arisaka-gg.org/p1-kk/.
Relevance. The collider companion of the Predictions Branch, and the sharpest statement in the cohort of the contrast this paper's Part III turns into theorems: the non-spatial circle versus Kaluza–Klein's spatial one, with the observable consequence (no angular satellites at the lock cluster) that makes non-spatiality falsifiable rather than rhetorical. Its front matter, course-style appendices, and problem-set discipline are also this paper's structural template. First cited in §8.1.

Arisaka, K. (2026). *Why Three Generations: A Three-Leg Derivation of $N_g = 3$ from BI-6, MSMF, and GDOS Morse Stability*. Arisaka-GG-P2-N3 preprint.
Preprint, available at https://preprints.arisaka-gg.org/p2-n3/.
Relevance. The neutrino-count companion of the Predictions Branch: the two-legged derivation of exactly three generations, whose CP-combinatorics leg — a physical CP phase requires $N \geq 3$ — ties the existence of the chiral loop holonomies of T-P4-5 to the family count itself. Cited in the cohort map as the sister paper whose subject matter meets this one at the flavor-holonomy seam. First cited in §8.1.

Arisaka, K. (2026). *The Electron as the Ground State of Nature: The Six Oldest Questions about the Electron, Closed and Fenced within GG-Theory*. Arisaka-GG-P3-Electron preprint.
Preprint, available at https://preprints.arisaka-gg.org/p3-electron/.
Relevance. The electron companion of the Predictions Branch: the ground-state reading of the electron's identity, stability, and pointlikeness. It meets this paper at the dial: the electron is the paradigm charged sector, its phase the canonical instance of the total dial angle $\theta + (p \cdot x - Et)/\hbar$ whose two-source decomposition is T-P4-3's subject, and its worked treatment in the parent program is the fullest single-particle illustration of the packaging this paper derives. First cited in §8.1.

Arisaka, K. (2026). *The Mirrors of Matter: Why Every Particle Has a Twin, Why Nature's Mirrors Break, and Why the Universe Is Made of Matter*. Arisaka-GG-P5-CPT preprint.
Preprint, available at https://preprints.arisaka-gg.org/p5-cpt/.
Relevance. The partner paper of the Foundations Branch and the GG-A9-SM–GG-A10-Cosmos interface of the cohort map: C, P, and T derived as orientation reversals of the three real bulk factors, the CPT theorem as total reversal, the chiral-pentagon CP phases, and the origin of the cosmic matter excess. It consumes this paper's derived complex structure — one shared J for the whole tower, carried down the map's shared-i edge — and extends the custody principle from the sectors to the mirrors. First cited in §8.1.

Arisaka, K. (2026). *The Proton as the Ground Geometric Attractor of Matter: The Eleven Questions of the Proton, Closed and Fenced within GG-Theory*. Arisaka-GG-P6-Proton preprint.

Preprint, available at https://preprints.arisaka-gg.org/p6-proton/.

Relevance. The fourth member of the Predictions Branch. It matters here because it is the cohort's heaviest use of the same object this paper derives the gauge phase from: baryon number is carried as a winding of the compact circle rather than as a bookkeeping label, so proton stability is a statement about the circle's topology and not about a symmetry imposed on top of it. Read beside T-P4-1 and the winding integrality of the ledger, it is the physical case in which the circle's compactness does work that no continuous phase could do. First cited in §8.1.

Arisaka, K. (2026). *The Kaxion: Cold Dark Matter as the Non-Spatial Compact-Angle Pseudoscalar of GG-6.* Arisaka-GG-C1-Kaxion preprint.
Preprint, available at https://preprints.arisaka-gg.org/c1-kaxion/.

Relevance. The dark-matter companion of the Observations Branch: the Kaxion, with its derived mass and laboratory window. It touches this paper at the compact-angle sector: the axion-like field is the cohort's other physical exploitation of the phase circle's topology, and its strong-CP context is the one place where a would-be CP angle is dynamically relaxed rather than lattice-locked — the complementary fate to the pentagonal phases of T-P4-5. First cited in §8.1.

Arisaka, K. (2026). *The Hubble Tension in the Light of GG-Theory: The Hubble Constant as One Reading of a Derived Expansion History $H(t)$.* Arisaka-GG-C2-Hubble preprint.
Preprint, available at https://preprints.arisaka-gg.org/c2-hubble/.

Relevance. The Hubble companion of the Observations Branch: the two-ends-of-one-corridor reading of the expansion-rate measurements. It enters this paper through the cosmological-dictionary interface: the comoving observer's particle inventories and effective descriptions are port-indexed boundary data in exactly the sense of §20, and the corridor analysis is the cohort's fullest exercise of that observer-indexed bookkeeping. First cited in §8.1.

Arisaka, K. (2026). *Black-Hole Entropy under GG-Theory: The GDOS Reading of the Horizon as an OGGEP Port — with the Exact Bekenstein–Hawking $A/4G_N$ Derived from Induced Gravity.* Arisaka-GG-C3-BH preprint.
Preprint, available at https://preprints.arisaka-gg.org/c3-bh/.

Relevance. The black-hole companion of the Observations Branch, and the source of the horizon-as-port reading used directly in §39 and Worked example IV: the six-dimensional bulk state pure and unitary, the horizon a locus of constitutional tracing, Hawking radiation the port's output, and the Bekenstein–Hawking entropy the entanglement of the traced sector. The stationary-exterior uniqueness and the Kerr residue discussed at objection O-P4-30 live against its backdrop. First cited in §8.1.

Arisaka, K. (2026). *Symmetry, Conservation, and Geometry: Seven Steps from Noether, Bianchi, to Hopkins–Singer, and the Ledger the Open-System Turn Pays Out.* Arisaka-GG-W1-Symmetry preprint.
Preprint, available at https://preprints.arisaka-gg.org/w1-symmetry/.

Relevance. The first of the two white papers, and the diachronic one: it follows a century of symmetry and conservation in the order the results were found, and watches the conviction that symmetry causes conservation loosen at seven successive steps until what remains conserved is an integer with nowhere to move. That endpoint is where this paper's winding integrality already stands, which makes W1 the historical account of why a topological charge, and not a Noether charge, is what the compact circle delivers. It derives nothing new; every result is carried at its parent's declared grade. First cited in §8.1.

Arisaka, K. (2026). *The Circle That Remembers, the Interval That Forgets: GG-Theory and the Sixteen Problems of Particle Physics and Cosmology.* Arisaka-GG-W2-Holonomy preprint. Preprint, available at https://preprints.arisaka-gg.org/w2-holonomy/.
Relevance. The second of the two white papers, and the closest of the three to this one. Its single exact statement — a holonomy requires a closed loop, and the interval fiber has none — is the asymmetry between the two non-spatial dimensions that lets the phase circle carry a gauge phase at all, and therefore lets T-P4-1 be a derivation rather than a redescription. W2 takes that statement to sixteen of the questions physics has not closed, and names the dated measurements on which six of the sixteen treatments die outright. Like its sibling it derives nothing new. First cited in §8.1.

B. Other references

Y Aharonov and D Bohm. Significance of electromagnetic potentials in the quantum theory. *Physical Review* **115** (1959) 485–491.
Relevance. The experiment-defining prediction that the electromagnetic phase is physical precisely and only through holonomies: a charged beam encircling confined flux acquires a measurable shift with no local field en route. For this paper it is the empirical cornerstone of the entire gauge half: the demonstration that the dial's differences are physics while its absolute setting is not — the fact that T-P4-1 geometrizes and that charge superselection generalizes. First cited in §3.

J-R Argand. Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques. *Paris (1806).*
Relevance. The geometric interpretation that domesticated the impossible numbers: complex quantities as directed magnitudes in a plane, with i the quarter-turn. On this paper's account Argand drew, without knowing it, the first portrait of the fiber plane (a, b) of Proposition 10.3 — the geometry was right; only the physical referent (a circle nature carries, a rotation time performs) was missing. First cited in §2.

A Ashtekar and A Magnon. Quantum fields in curved space-times. *Proceedings of the Royal Society of London A* **346** (1975) 375–394.
Relevance. With Kay (1978), the uniqueness of the positive-energy complex structure on a stationary arena — the selection theorem behind §26. First cited in §26.

S Bachmann, D-A Deckert, and A Pizzo. The mass shell of the Nelson model without cut-offs. *Journal of Functional Analysis* ***263*** *(2012) 1224–1282.*

Relevance. Theorem-grade control of the massless Nelson model's dressed one-particle structure with the ultraviolet cutoff removed. Together with Pizzo (2003) it supplies the existence half of the door theorem's middle rung (T-P4-9): the dressed states whose soft clouds the theorem classifies as port assignments do exist, with the regularity the sector analysis needs, and with no cutoff artifact underwriting the conclusion. First cited in §38.

M V Berry. Quantal phase factors accompanying adiabatic changes. *Proceedings of the Royal Society of London A* ***392*** *(1984) 45–57.*

Relevance. The adiabatic phase: a quantum system transported slowly around a parameter loop returns with a phase fixed by the loop's geometry alone. Its role here is as the modern era's second demonstration (after Aharonov–Bohm) that quantum phases are holonomies — loop quantities of a real connection — the reading that T-P4-1 completes by promoting the fiber to a dimension. First cited in §4.

F Bloch and A Nordsieck. Note on the radiation field of the electron. *Physical Review* ***52*** *(1937) 54–59.*

Relevance. The original diagnosis of the infrared problem: soft-photon emission renders every scattering probability formally zero unless arbitrarily soft quanta are summed. The logarithmically divergent overlap exponent of (35) — the mechanism by which velocity sectors become orthogonal — is this paper's real-presentation form of their signature, and the boundary the door theorem T-P4-9 now crosses at the model level. First cited in §38.

R Bombelli. L'Algebra. *Bologna (1572).*

Relevance. The book that made the impossible numbers unavoidable: for the casus irreducibilis of the cubic, every algebraic route to the real roots passes through complex intermediaries. Real in, real out, complex in between — the exact arc this paper finds in physics (real bulk, complex amplitudes, real records), discovered as a fact about polynomials four and a half centuries before it was located in nature. First cited in §2.

M Born. Zur Quantenmechanik der Stossvorgänge. *Zeitschrift für Physik* ***37*** *(1926) 863–867.*

Relevance. The probability interpretation: $|\psi|^2$ as the measurable content of the wavefunction, with the phase left physical only through interference. In this paper's terms Born's rule is the statement that registration marginalizes the unreadable dial while relative dial angles survive — the operational face of the fiber's unreadability, whose square-modulus form the parent program derives via Gleason. First cited in §4.

D Buchholz. Collision theory for massless bosons. *Communications in Mathematical Physics* **52** (1977) 147–173.

Relevance. The rigorous collision theory for massless bosons: asymptotic fields and scattering states constructed without a mass gap, under general locality and spectrum assumptions. In this paper it is one of the two footholds of the infrared boundary (§38): it secures the massless quanta's own asymptotic structure, so that the corner T-P4-9 then closes at the model level is the charged infraparticle's dressing, not the photon sector itself. First cited in §38.

D Buchholz. Gauss' law and the infraparticle problem. *Physics Letters B* ***174*** *(1986) 331–334.*
Relevance. The structural reading of the infraparticle: by Gauss' law the asymptotic electromagnetic flux is a superselected observable, so a charged particle's velocity is written in its cloud at infinity and sharp mass shells are excluded. The door theorem adopts this sector structure wholesale and adds the framework's placement — the flux label is dictionary data at AX-3, a port assignment, and the derived complex structure is blind to it (T-P4-9). The flux observable is what makes the sector label *physical* rather than notational. First cited in §38.

G Cardano. Artis Magnae, sive de Regulis Algebraicis (Ars Magna). *Nuremberg (1545).*
Relevance. The book where the square roots of negatives first earned their keep, in the solution of the cubic, accompanied by their author's famous disparagement. The starting point of the five-century chronology of §6: an invention explicitly labeled useless at birth, whose eventual physical necessity became the puzzle this paper answers. First cited in §2.

A-L Cauchy. Mémoire sur les intégrales définies, prises entre des limites imaginaires. *Paris (1825).*
Relevance. The memoir that revealed the rigidity of the complex domain: integrals between imaginary limits, contour independence, and the residue calculus. That rigidity — one derivative implying all, values in a region determining values everywhere — is what this paper's bridge traces to one-sided frequency support: the analytic miracle as the fingerprint of an orientation of time. First cited in §2.

T Chen, J Fröhlich, and A Pizzo. Infraparticle scattering states in non-relativistic QED: I. The Bloch–Nordsieck paradigm. *Communications in Mathematical Physics* ***294*** *(2010) 761–825.*
Relevance. The rigorous construction of infraparticle scattering states in nonrelativistic QED, exhibiting velocity-superselected soft clouds as the correct asymptotic objects. This is the scattering-level import of the door theorem's middle rung: the states exist, their clouds carry the sector label of Lemma 38.3, and the dressed asymptotic dynamics has exactly the free-plus-c-number form whose polar phase T-P4-9 reads. The strongest rigorous result standing under the door. First cited in §38.

G Chiribella, G M D'Ariano, and P Perinotti. Informational derivation of quantum theory. *Physical Review A* ***84*** *(2011) 012311.*
Relevance. The informational derivation of quantum theory: complex QM singled out among operational theories by axioms about information processing (purification chief among them). Cited at objection O-P4-10 as convergent evidence of a different kind from this paper's: the reconstructions prove it had to be $\mathbb{C}$; the geometric account exhibits what $\mathbb{C}$ was. First cited in §X.

V Chung. Infrared divergence in quantum electrodynamics. *Physical Review* ***140*** *(1965) B1110–B1122.*
Relevance. The coherent-state resolution of the infrared divergences: charged states dressed with soft-photon clouds of definite classical profile, rendering transition probabilities finite. Its dressed

sectors are exactly the coherent displacements of Lemma 38.1 — sector-moving, J-preserving — and its construction anchors the port reading of the infrared in §38. First cited in §38.

P A M Dirac. The Principles of Quantum Mechanics. *Clarendon Press, Oxford (1930).*
Relevance. The book that fixed quantum mechanics' modern form, transformation theory and all, weaving the i into the commutation relations themselves. Cited in the history as the moment the complex unit became load-bearing formalism rather than removable shorthand — the state of affairs this paper's two-source derivation is designed to explain rather than overturn. First cited in §4.

M G Eastwood, R Penrose, and R O Wells. Cohomology and massless fields. *Communications in Mathematical Physics* **78** (1981) 305–351.
Relevance. The rigorous Penrose transform: massless fields of helicity h as $H^1(\mathbb{PT}, \mathcal{O}(-2h-2))$, with positive frequency supported on $\mathbb{PT}^+$ — the third level of the bridge. First cited in §5.

L Euler. Introductio in analysin infinitorum. *Lausanne (1748).*
Relevance. The Introductio, home of $e^{i\theta} = \cos\theta + i\sin\theta$: the exponential bound to the circle. On this paper's reading Euler's identity is not a curiosity but the entire mechanism in miniature — the harmonic analysis of a compact real dimension *is* the complex exponential, which is why a physical circle makes complex numbers appear wherever charge lives (T-P4-1). First cited in §2.

L D Faddeev and P P Kulish. Asymptotic conditions and infrared divergences in quantum electrodynamics. *Theoretical and Mathematical Physics* **4** (1970) 745–757.
Relevance. The dressed-state reformulation of QED's asymptotic dynamics: charged particles paired with their soft-photon clouds via an asymptotic interaction that never switches off, yielding infrared-finite S-matrix elements between coherent-dressed states. Here it organizes the charged sectors of §38 and anchors the top rung of the door theorem T-P4-9 — the dressing is the sector label, Lemma 38.1 shows it never touches J, and the construction's full rigor is the theorem's named frontier. First cited in §38.

J Fourier. Théorie analytique de la chaleur. *Paris (1822).*
Relevance. The analytical theory of heat, and with it the decomposition of arbitrary signals into exponentials — the tool that trained physics to complexify, compute, and discard. Cited in §3 as the first wave of the quiet infiltration: the i as pure bookkeeping, a habit whose eventual failure (in quantum mechanics) became the puzzle. First cited in §3.

A Fresnel. Mémoire sur la loi des modifications que la réflexion imprime à la lumière polarisée. *Mémoires de l'Académie des Sciences (1823).*
Relevance. Complex amplitudes for polarized light: the first sustained physical use of complex quantities, with phase and amplitude carried in one symbol and the imaginary part discarded at the end. The optics of the 1820s is where the phasor habit began, and this paper's history places it as the template for eighty years of removable i. First cited in §3.

J Fröhlich. On the infrared problem in a model of scalar electrons and massless, scalar bosons. *Annales de l'Institut Henri Poincaré A* ***19*** *(1973) 1–103.*

Relevance. The founding rigorous analysis of the infraparticle: in the massless Nelson model the fixed-momentum Hamiltonian has no ground state in Fock space, and the physical states live in velocity-indexed coherent sectors. Lemma 38.3 is the quantitative skeleton of Fröhlich's phenomenon — the logarithmically divergent cloud and the power-law sector orthogonality — and the door theorem's middle rung stands on the existence theory this paper opened. First cited in §38.

D Gabor. Theory of communication. *Journal of the IEE **93** (1946) 429–457.*
Relevance. The theory of communication, and the analytic signal: a real waveform completed by its Hilbert-transform quadrature into a complex signal whose modulus is the envelope and whose argument is the instantaneous phase. Cited at Worked example I as the engineer's form of T-P4-6 — the construction this paper shows to be, in the quantum context, not a trick but the boundary value of forward-tube holomorphy. First cited in §35.1.

C F Gauss. Theoria residuorum biquadraticorum, commentatio secunda. *Göttingen (1831).*
Relevance. The paper that made the plane official and coined the very name "complex numbers," ending three centuries of ontological embarrassment by geometric fiat. In the chronology it marks the completion of the domestication — and the beginning of the deeper puzzle, since a geometric interpretation within mathematics still leaves open what, in nature, the plane is a picture of. First cited in §2.

A M Gleason. Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics **6** (1957) 885–893.*
Relevance. The measure-theoretic theorem: every countably additive probability assignment on the closed subspaces of a Hilbert space of dimension three or more is of trace form. In the parent program it fixes the square-modulus form of the Born weights; cited here at the actualization layer (§21) as the reason the form of the probabilities, unlike their per-run values, is theorem-grade. First cited in §21.

M B Green and J H Schwarz. Anomaly cancellations in supersymmetric $D = 10$ gauge theory and superstring theory. *Physics Letters B* **149** (1984) 117–122.
Relevance. The anomaly-cancellation result that fixes the string program's dimension and gauge structure — cited in the three-roads comparison (§43) as the location of that program's irreducible premises. First cited in §43.

R Haag. Local Quantum Physics: Fields, Particles, Algebras. *Springer, Berlin* (1992).
Relevance. The standard treatise of algebraic quantum field theory. Two of its chapters are load-bearing here: the Haag–Ruelle scattering theory that powers the interacting completion T-P4-8, and the superselection theory whose abstract sector analysis the three-superselections pattern of §39 instantiates geometrically — charge, horizon, and infrared sectors as three unreadabilities with one lesson. First cited in §37.

W R Hamilton. Theory of conjugate functions, or algebraic couples. *Transactions of the Royal Irish Academy **17** (1837) 293–422.*

Relevance. The algebraic couples: $\mathbb{C}$ rebuilt as ordered pairs of reals with a stipulated multiplication — the first real presentation, and the proof that i is eliminable in principle. This paper stands to Hamilton as physics to his algebra: the theorems of Part IV supply the selection his stipulation lacked — which quarter-turn, why shared, why unique — turning a philosophical nicety into a statement about nature. First cited in §2.

L Hardy. Quantum theory from five reasonable axioms. *arXiv:quant-ph/0101012 (2001).*
Relevance. The five reasonable axioms: the paper that opened the modern operational-reconstruction program, deriving quantum theory's structure from information-theoretic principles. Cited at objection O-P4-10 within the convergence argument: reconstruction locates the necessity of the complex formalism; the present account locates its physical referent. First cited in §X.

S W Hawking. Particle creation by black holes. *Communications in Mathematical Physics **43** (1975) 199–220.*
Relevance. Particle creation by black holes: the thermal flux at temperature $\kappa/2\pi$ that made horizon thermodynamics physical. Worked example IV computes its structural core in four lines of covariance algebra — the traced half of the horizon's squeezed pairing is exactly Planckian — and the port reading of §39 places the result where the parent program's dictionary says it belongs. First cited in §35.4.

P Jordan, J von Neumann, and E Wigner. On an algebraic generalization of the quantum mechanical formalism. *Annals of Mathematics **35** (1934) 29–64.*
Relevance. The algebraic classification of quantum formalisms: the finite-dimensional Jordan algebras admitting a quantum-mechanical interpretation, with the real, complex, and quaternionic families delimited. The classical ancestor of every "why complex?" discussion, cited at objection O-P4-10; on this paper's account its complex branch is the one nature instantiates because a circle and a clock source a shared J. First cited in §X.

B S Kay. Linear spin-zero quantum fields in external gravitational and scalar fields. *Communications in Mathematical Physics* **62** (1978) 55–70.
Relevance. Kay's uniqueness theorem for the stationary one-particle structure — the second half of the selection theorem of §26. First cited in §26.

B S Kay and R M Wald. Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon. *Physics Reports **207** (1991) 49–136.*
Relevance. The uniqueness and thermality theorems for quasifree states on spacetimes with bifurcate Killing horizons — including the obstruction that no Hadamard state on Kerr is invariant under the horizon-generating flow. Cited at objection O-P4-30 and §49 as the honest boundary of the stationary-case resolution: the rotating anchor is physics, and it is open. First cited in §X.

M-C Li *et al.*. Testing real quantum theory in an optical quantum network. *Physical Review Letters **128** (2022) 040402.*
Relevance. One of the 2022 experimental realizations of the network Bell test that excludes real quantum mechanics with real tensor products. Cited beside the theoretical exclusion in §4:

the shared-i mandate is not a thought experiment but a laboratory result, and T-P4-4 is the structure that satisfies it by construction. First cited in §4.

L J Mason and N M J Woodhouse. Integrability, Self-Duality, and Twistor Theory. *Oxford University Press (1996).*
Relevance. The monograph connecting twistor constructions to integrable systems: self-duality as the master integrability, with the Ward correspondence as its engine. Cited at objection O-P4-1 as the direction in which the nonlinear conjecture points: if the nonlinear twistor geometry is the integrable packaging of the two real sources, this is the mathematics in which the packaging would be exhibited. First cited in §X.

R E A C Paley and N Wiener. Fourier Transforms in the Complex Domain. *American Mathematical Society Colloquium Publications* **19** (1934).
Relevance. The classical source of the Paley–Wiener theorem (Theorem 13.3): one-sided frequency support is equivalent to Hardy-space holomorphy — the first level of the bridge. First cited in §13.

R Penrose. Twistor algebra. *Journal of Mathematical Physics* **8** (1967) 345–366.
Relevance. The founding paper of twistor theory: the algebra of twistors, the incidence correspondence, and the proposal that light rays rather than events are the primitive objects. For this paper it defines the geometric stage of Course B and the object — the division of $\mathbb{PT}$ — whose identification with the causal-positive frequency splitting (T-P4-7) is the central result offered to its school. First cited in §5.

R Penrose. Solutions of the zero-rest-mass equations. *Journal of Mathematical Physics **10** (1969) 38–39.*
Relevance. The contour-integral form of the massless field representation: solutions of the zero-rest-mass equations as integrals of holomorphic functions over lines in twistor space. The original transform, executed concretely in Worked example II and Lecture B.5, where its homogeneity counting and pole placement become this paper's statement that frequency sign is contour topology. First cited in §14.

R Penrose. Nonlinear gravitons and curved twistor theory. *General Relativity and Gravitation* **7** (1976) 31–52.
Relevance. The nonlinear graviton construction, and with it the "googly" problem — the twistor program's named central hurdle, recorded in the comparison table as open on both sides. First cited in §45.

R Penrose and M A H MacCallum. Twistor theory: an approach to the quantization of fields and space-time. *Physics Reports* **6** (1973) 241–315.
Relevance. The foundational exposition, including the correspondence between positive-frequency massless fields and holomorphic data on $\mathbb{PT}^+$ — the component of the bridge that makes the identification of §33 available. First cited in §5.

R Penrose and W Rindler. Spinors and Space-Time, Volume 2: Spinor and Twistor Methods in Space-Time Geometry. *Cambridge University Press* (1986).

Relevance. The standard reference for twistor geometry — the incidence relation, the helicity form, and the $\mathbb{PT}^{\pm}$ division used in Definition 14.1. First cited in §14.

R Penrose. The Road to Reality: A Complete Guide to the Laws of the Universe. *Jonathan Cape, London* (2004).

Relevance. The canonical statement of the complex-number question this preprint answers in calibrated form. First cited in §5.

A Pizzo. One-particle (improper) states in Nelson's massless model. *Annales Henri Poincaré **4** (2003) 439–486*.

Relevance. The iterated-dressing construction of the massless Nelson model's one-particle states, with Hölder regularity in the total momentum. This is the constructive engine behind the door theorem's middle rung: it builds exactly the dressed states whose coherent clouds T-P4-9 classifies as port assignments, and on which the sector-blindness of the derived J (Lemma 38.4) is then a theorem rather than a hope. First cited in §38.

M-O Renou, D Trillo, M Weilenmann, T P Le, A Tavakoli, N Gisin, A Acín, and M Navascués. Quantum theory based on real numbers can be experimentally falsified. *Nature* **600** (2021) 625–629.

Relevance. The experimental mandate for a single shared complex structure — satisfied here by construction from one circle and one causal order. First cited in §4.

B Riemann. Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse. *Inauguraldissertation, Göttingen (1851)*.

Relevance. The inaugural dissertation: complex function theory refounded geometrically, with the surfaces that bear his name. In the chronology it marks the maturation of the rigidity discovered by Cauchy — and the final form of the mathematics whose physical referent (an orientation of time, conformally completed) the bridge of Part V identifies. First cited in §2.

E Schrödinger. Quantisierung als Eigenwertproblem (Vierte Mitteilung). *Annalen der Physik **81** (1926) 109–139*.

Relevance. The fourth communication: the paper in which the wavefunction's complexity becomes unavoidable, against its author's own sustained effort to keep ψ real. Cited in §4 as the founding instance of the puzzle — the first i in physics that would not cancel — whose resolution (the shared quarter-turn of the clock, not a mode-by-mode shorthand) is T-P4-2. First cited in §4.

M P Solèr. Characterization of Hilbert spaces by orthomodular spaces. *Communications in Algebra **23** (1995) 219–243*.

Relevance. The lattice-theoretic characterization: infinite orthomodular spaces with the relevant completeness are Hilbert spaces over the three classical division rings. Cited at objection O-P4-10 among the reconstruction results that narrow the field to real, complex, or quaternionic — the trichotomy within which the experiments and the shared-J theorem of T-P4-4 then select. First cited in §X.

C P Steinmetz. Complex quantities and their use in electrical engineering. *Proceedings of the International Electrical Congress, Chicago (1893) 33–74*.

Relevance. The address that industrialized the phasor: alternating-current engineering rebuilt on complex quantities, with the imaginary part discarded at the end of every calculation. The paradigm of removable i — and thereby, on this paper's telling, the trap: eighty years of training that left physics without the vocabulary to say what changed in 1926. First cited in §3.

M H Stone. On one-parameter unitary groups in Hilbert space. *Annals of Mathematics **33** (1932) 643–648*.

Relevance. The one-parameter unitary groups theorem: continuous unitary evolutions are exactly e^{itH} with self-adjoint H. Cited at objection O-P4-9, where its role is clarified: Stone interlocks i, t, and H *within* complex quantum mechanics; the present account supplies the prior fact — why the state space is complex and what selects its J — after which Stone's theorem becomes derived scaffolding. First cited in §X.

R F Streater and A S Wightman. PCT, Spin and Statistics, and All That. *W. A. Benjamin, New York* (1964).

Relevance. The classic axiomatic treatment: Wightman functions, the forward-tube analyticity that this paper reads as the field-theoretic form of causal positivity (§32), and the PCT theorem whose orientation-bookkeeping counterpart is T-P4-5. It is the bridge's middle level in its original habitat, and the reading guide points the student to its chapters as Course A's rigorous companion. First cited in §32.

E C G Stueckelberg. Quantum theory in real Hilbert space. *Helvetica Physica Acta **33** (1960) 727–752*.

Relevance. The serious real-Hilbert-space program: quantum theory over $\mathbb{R}$, with the complex unit replaced by a superselected operator J. Cited in §4 as the road not taken's most rigorous milestone — its single-system half survives (and anticipates the real-presentation spirit), while its composite structure is what the network experiments later excluded, exactly at the unshared tensor product. First cited in §4.

E C Titchmarsh. Introduction to the Theory of Fourier Integrals. *Oxford University Press, 2nd edition (1948)*.

Relevance. The classical treatise on Fourier integrals: conjugate functions, the Hilbert transform's L^2 theory, and the Paley–Wiener circle of ideas in their standard form. The reading guide's analysis shelf points Course A's student here for every ϵ the lectures wave through; the theorems used in this paper live in its chapters V and VI. First cited in §63.2.

W G Unruh. Notes on black-hole evaporation. *Physical Review D **14** (1976) 870–892*.

Relevance. The acceleration temperature: a uniformly accelerated detector in the Minkowski vacuum thermalizes at $a/2\pi$, via the pairing of Rindler wedges. Worked example IV's computation is its structural skeleton, and the port reading — which modes an instrument counts is dictionary data — is where this paper files the effect's famous observer-dependence. First cited in §35.4.

J von Neumann. Mathematische Grundlagen der Quantenmechanik. *Springer, Berlin (1932).*
Relevance. The mathematical foundations: the complex separable Hilbert space installed as quantum mechanics' first postulate, where every textbook since has kept it. Cited at objection O-P4-41 as the formulation this paper does not overturn but sources: the postulate stands, and acquires a derivation with named hypotheses and named residues. First cited in §X.

R M Wald. Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics. *University of Chicago Press (1994).*
Relevance. The standard exposition of quantum field theory in curved spacetime: one-particle structures, quasifree states, and the algebraic formulation, in the form this paper's apparatus imports. Cited at objection O-P4-11, which states exactly what is taken (the machinery) and what is added (the constitutional arena, the circle–clock pairing, the assembly). First cited in §X.

R S Ward. On self-dual gauge fields. *Physics Letters A **61** (1977) 81–82.*
Relevance. The self-dual correspondence: anti-self-dual gauge fields encoded as holomorphic bundles over twistor space — with the nonlinear graviton, the engine of twistor theory's nonlinear era. Cited at §49 as the frontier the bridge has not reached: the conjecture that these constructions are the integrable packaging of the two real sources is recorded, and open, at objection O-P4-1. First cited in §49.

H Weyl. Gravitation und Elektrizität. *Sitzungsberichte der Preussischen Akademie der Wissenschaften* (1918) 465–480.
Relevance. Gravitation and electricity: the original gauge theory, in which local *scale* invariance was to unify gravitation and electromagnetism. It failed physically (Einstein's measuring-rod objection) and succeeded structurally: its 1929 phase reinterpretation created modern gauge theory, and the parent program's two non-spatial dimensions — scale and phase — are, in a precise sense, both of Weyl's ideas made geometry at last. First cited in §3.

H Weyl. Elektron und Gravitation. I. *Zeitschrift für Physik* **56** (1929) 330–352.
Relevance. The birth of the modern gauge principle: the 1918 scale theory reinterpreted as local phase invariance, putting $e^{i\lambda(x)}$ into electrodynamics and, unknowingly, opening the question this paper closes — what the phase is an angle *of*. On the present account Weyl's phase is the circle's coordinate, his gauge freedom the dial re-zeroing of T-P4-5, and his 1929 insight the second wave of the quiet infiltration traced in §3. First cited in §3.

G C Wick, A S Wightman, and E P Wigner. The intrinsic parity of elementary particles. *Physical Review **88** (1952) 101–105.*
Relevance. The paper that introduced superselection rules: the observation that relative phases between certain sectors of the Hilbert space are unmeasurable in principle, so the superposition principle holds only within sectors. The custody principle (Proposition 39.1) is this paper's geometric completion of that observation for the three instances the derivation meets: each superselected phase is identified with a global real datum — a winding, a horizon interior, a cloud at infinity — owned by the operational dictionary, while the derived complex structure is proved identical across sectors. First cited in §39.

E Witten. Perturbative gauge theory as a string theory in twistor space. *Communications in Mathematical Physics* **252** (2004) 189–258.

Relevance. The paper that returned twistor variables to the center of perturbative gauge theory: gauge-theory amplitudes as a string theory in twistor space, igniting the modern amplitudes program. Cited in the three-roads comparison and at objection O-P4-6 as the twistor column's contemporary strength on interactions — and as the natural interface at which an amplitudes-level extension of the bridge would attach. First cited in §45.

T T Wu and C N Yang. Concept of nonintegrable phase factors and global formulation of gauge fields. *Physical Review D* **12** (1975) 3845–3857.

Relevance. The bundle formulation that recognized the gauge phase as holonomy — the precedent for reading the phase geometrically, completed here by promoting the fiber to a real dimension. First cited in §3.